STATISTICAL INFERENCE ON STRUCTURED HIGH-DIMENSIONAL MODELS USING LIKELIHOOD-BASED METHODS

by

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Abstract

In contemporary statistics, datasets are typically collected with high-dimensionality that originate from high-dimensional statistical models, where the dimension of the parameter space can be comparable or even significantly greater than the sample size. For these high-dimensional data, feasible statistical inference cannot be made without additional structural assumptions. One of the predominant collection of statistical inference methods for varieties of structured high-dimensional models is based on spectral methods, such as spectral clustering for stochastic block models, or penalized spectral methods, such as sparse principal component analysis. In contrast, likelihood-based methods for such non-classical statistical models are relatively under-explored.

This dissertation aims to develop easy-to-implement likelihood-based inference methods for certain structured high-dimensional statistical models and the corresponding theoretical understanding of these methods. The first major contribution of this dissertation is on the development of a novel matrix shrinkage prior for Bayesian estimation of jointly sparse spiked covariance matrices in high dimensions. The spiked covariance matrix is reparameterized in terms of the latent factor model, where the loading matrix is assigned a novel matrix shrinkage spike-and-slab LASSO prior. We study the posterior contraction rate of the principal subspace with respect to the two-to-infinity norm loss, a novel loss function measuring the distance between subspaces that is able to capture element-wise eigenvector deviations.

The second contribution of this dissertation is on the development of likelihood-based inference methods for the random dot product graph model. Both the global

estimation and local estimation are considered. For the global estimation task, the minimax lower bound is established, and this minimax lower bound is achieved by a Bayesian method, referred to as the posterior spectral embedding. We also designed a handy Metropolis-Hastings sampler for convenient computation of the posterior inference. For the local estimation task, we first define the local efficiency rigorously and then propose a novel one-step procedure that takes advantage of the derivatives information of the likelihood function of the graph model. Furthermore, we establish the local efficiency of the proposed one-step estimator. In contrast, the previously widely adopted spectral-based adjacency spectral embedding method is proven to be locally inefficient.

The content of this dissertation corresponds to one paper that has been accepted for publication on Biometrika and two preprints, namely, the works [1–3].

Dissertation Readers

Primary Reader and Advisor: Yanxun Xu

Second Reader: Carey E. Priebe

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Chapter 1

Introduction

1.1 A gentle start and overview

Statistical problems that are related to high-dimensional datasets have been a heated topic not only within the field of statistics but also pervasive in machine learning, as well as a variety of application domains. Roughly speaking, high-dimensional data arise when the number of features can be comparable or even significantly larger than the sample size of the dataset. For example, in genomics studies, the number of genes is typically much larger than the number of subjects [4]. In computer vision, the number of pixels in each image can be comparable to or exceed the number of images when the resolution of these images is relatively high [5, 6]. To work with these challenging high-dimensional datasets, statisticians typically impose certain practically meaningful structural assumptions on the underlying statistical model, such that feasible statistical analysis can be carried out. This dissertation aims to address statistical inference tasks for certain structured high-dimensional models using likelihood-based methods and provide theoretical insight into these methods. Specifically, we focus on the following two sampling models as interesting representatives of structured high-dimensional models and develop the corresponding likelihood-based methods for statistical inference: the sparse spiked covariance matrix models and the random dot product graph model.

When dealing with featured high-dimensional datasets, covariance matrix estimation plays a central role in understanding the complex structure of the data and has received significant attention in various contexts, including latent factor models [7, 8], Gaussian graphical models [9, 10], etc. However, in the high-dimensional setting, additional structural assumptions are often necessary in order to address challenges associated with statistical inference [11]. For example, sparsity is introduced for sparse covariance/precision matrix estimation [12–14], and a low-rank structure is enforced in spiked covariance matrix models [15, 16]. Readers can refer to [12] for a recent literature review. Chapter 2 of this dissertation focuses specifically on the likelihood-based inference on sparse spiked covariance matrix model in high dimensions using Bayesian shrinkage prior.

Another type of high-dimensional data is network data. Using networks/graphs as a data structure to represent network data with the vertices denoting entities and the edges encoding relationships between vertices, has become increasingly important in a broad range of applications, including social networks [17], brain imaging [18], and neuroscience [19, 20]. For example, in a Facebook network, vertices represent users, and the occurrence of an edge linking any two users indicates that they are friends on Facebook. When one collects random graph data, it may be costly or even infeasible to collect individual-specific attributes that are heterogeneous across individuals, while only the adjacency matrix of the graph is accessible. For example, in studying the structure of a Wikipedia page network, collecting the hyperlinks among articles is much more feasible than collecting the attributes associated with the individual articles. To model the unobserved vertex-specific attributes that result in the observed network, the authors of [21] proposed latent positions graphs, in which each vertex is associated with an unobserved Euclidean vector called the latent position, and the edge probability between any two vertices only depends on their latent positions. There is vast literature addressing statistical inference on latent positions graphs. For an incomplete list of references, see [22–26], among others. The specific latent position graph model considered in this dissertation is the random dot product graph model, an architecturally simple yet useful and flexible low-rank random graph model, and Chapters 3 and 4 provide two inference perspectives for this graph model using different likelihood-based methods.

The rest of the dissertation is organized as follows. The remaining part of Section 1 provides necessary mathematical notations, the formal introduction of the sparse spiked covariance matrix model and the random dot product graph model of interest, and the corresponding necessary background. Chapter 2 focuses on statistical inference on the sparse spiked covariance matrix model with high dimensionality. In particular, we develop a novel matrix shrinkage prior, referred to as the matrix spiked-and-slab LASSO prior, for Bayesian estimation of the sparse spiked covariance matrix model with high dimensionality, and establish the corresponding convergence results. This chapter corresponds to the paper [1]. Chapters 3 and 4 address statistical inference tasks for random dot product graphs with different emphasis and different methods: Chapter 3 tackles the optimal global estimation task using a Bayesian approach with strong convergence guarantee, whereas a novel one-step estimator is designed to solve the problem of efficient local estimation in Chapter 4, together with an elegant asymptotic distribution result. The two papers involved are [2] and [3], respectively. Technical proofs are collected in Chapter 5.

1.2 Notations

Let p and r be positive integers. We adopt the shorthand notation $[p] = \{1, \dots, p\}$. For any finite set S, we use $|S|$ to denote the cardinality of S. The symbols $\lesssim$ and $\gtrsim$ mean the inequality up to a universal constant, *i.e.*, $a \lesssim b$ ($a \gtrsim b$, resp.) if $a \leq Cb$ ($a \geq Cb$) for some absolute constant $C > 0$. We write $a \asymp b$ if $a \lesssim b$ and $a \gtrsim b$. The $p \times r$ zero matrix is denoted by $\mathbf{0}_{p\times r}$, and the p-dimensional zero column vector is denoted by $\mathbf{0}_p$. When the dimension is clear, the zero matrix is simply denoted by $\mathbf{0}$. The $p\times p$ identity matrix is denoted by $\mathbf{I}_p$, and when the dimension is clear, is denoted by $\mathbf{I}$. The vector with all entries being 1 is denoted by the boldface $\mathbf{1}$. An orthonormal r-frame in $\mathbb{R}^p$ is a $p\times r$ matrix $\mathbf{U}$ with orthonormal columns, *i.e.*, $\mathbf{U}^{\mathrm{T}}\mathbf{U} = \mathbf{I}_{r\times r}$. The set of all orthonormal r-frames in $\mathbb{R}^p$ is denoted by $\mathbb{O}(p, r)$. When $p = r$, we write $\mathbb{O}(r) = \mathbb{O}(r, r)$. For a p-dimensional vector $\mathbf{x}\in\mathbb{R}^d$, we make the convention that x_j denotes its jth component for $j\in[d]$, and will use $\mathbf{x} = [x_1,\ldots,x_d]^{\mathrm{T}}$ to represent it as a column vector. For an integer p, $1\leq p\leq\infty$ and $\mathbf{x}\in\mathbb{R}^d$, we use $\|\mathbf{x}\|_p$ to denote its ℓ_p-norm, and when $p=\infty$, $\|\mathbf{x}\|_\infty = \max_{k=1,\ldots,d}|x_k|$. In particular, we drop the norm subscript p when $p = 2$, namely, $\|\mathbf{x}\| = \|\mathbf{x}\|_2$. For any two vectors $\mathbf{x} = [x_1,\ldots,x_d]^{\mathrm{T}}$ and $\mathbf{y} = [y_1,\ldots,y_d]^{\mathrm{T}}$ in $\mathbb{R}^d$, the vector inequality $\mathbf{x}\leq\mathbf{y}$ means that $x_k\leq y_k$ for all $k = 1,2,\ldots,d$. For a symmetric square matrix $\mathbf{\Sigma}\in\mathbb{R}^{p\times p}$, we use $\lambda_k(\mathbf{\Sigma})$ to denote the kth-largest eigenvalue of $\mathbf{\Sigma}$, and for any rectangular matrix $\mathbf{X}$, we use $\sigma_k(\mathbf{X})$ to denote its kth largest singular value. For a matrix $\mathbf{A}\in\mathbb{R}^{p\times r}$, we use $\mathbf{A}_{j*}$ or $(\mathbf{A})_{j*}$ to denote the row vector formed by the jth row of $\mathbf{A}$, $\mathbf{A}_{*k}$ or $(\mathbf{A})_{*k}$ to denote the column vector formed by the kth column of $\mathbf{A}$, the lower case letter a_{ij} to denote the (i, j)-th element of $\mathbf{A}$, $\|\mathbf{A}\|_{\mathrm{F}} = \sqrt{\sum_{j=1}^p\sum_{k=1}^r a_{jk}^2}$ to denote the Frobenius norm of $\mathbf{A}$, $\|\mathbf{A}\|_2 = \sqrt{\lambda_1(\mathbf{A}^{\mathrm{T}}\mathbf{A})}$ to denote the operator norm of $\mathbf{A}$, $\|\mathbf{A}\|_{2\to\infty} = \max_{\|\mathbf{x}\|_2=1}\|\mathbf{A}\mathbf{x}\|_\infty$ to denote the two-to-infinity norm of $\mathbf{A}$, and $\|\mathbf{A}\|_\infty = \max_{\|\mathbf{x}\|_\infty=1}\|\mathbf{A}\mathbf{x}\|_\infty$ to denote the (matrix) infinity norm of $\mathbf{A}$. The prior and posterior distributions appearing in this dissertation are all denoted by Π, and the densities of Π with respect to the underlying sigma-finite measure are denoted by π, unless otherwise specified. We say that a sequence of events $(E_n)_{n=1}^\infty$ occurs almost always, if $\mathbb{P}(\bigcup_{n=1}^\infty\bigcap_{k=n}^\infty E_k) = 1$. We use the shorthand notation $a\vee b = \max(a, b)$ and $a\wedge b = \min(a, b)$ for any $a, b\in\mathbb{R}$. For any two positive semidefinite matrices $\mathbf{\Sigma}_1$ and $\mathbf{\Sigma}_2$ of the same dimension, the notation $\mathbf{\Sigma}_1\preceq\mathbf{\Sigma}_2$ ($\mathbf{\Sigma}_1\succeq\mathbf{\Sigma}_2$) means that $\mathbf{\Sigma}_2-\mathbf{\Sigma}_1$ ($\mathbf{\Sigma}_1-\mathbf{\Sigma}_2$) is positive semidefinite, and we say that $\mathbf{\Sigma}_1$ is no greater (no less) than $\mathbf{\Sigma}_2$ in spectra.

1.3 Background on the sparse spiked covariance matrix model

Let us consider the sparse spiked covariance matrix models under the Gaussian sampling distribution as one representative of structured high-dimensional models. The spiked covariance matrix models, originally named in [16], is a class of models that can be described as follows: The observations $\mathbf{y}_1, \ldots, \mathbf{y}_n$ are independently drawn from the p-dimensional mean-zero normal distribution with covariance matrix $\mathbf{\Sigma}$ of the form

$$\mathbf{\Sigma} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^{\mathrm{T}} + \sigma^2 \mathbf{I}_p, \tag{1.1}$$

where $\mathbf{U}$ is a $p \times r$ matrix with orthonormal columns, $\mathbf{\Lambda} = \mathrm{diag}(\lambda_1, \cdots, \lambda_r)$ is an $r \times r$ diagonal matrix with $\lambda_1 \geq \ldots \geq \lambda_r > 0$, and $r < p$. Since the spectrum of the covariance matrix is $\{\lambda_1 + \sigma^2, \ldots, \lambda_r + \sigma^2, \sigma^2, \cdots, \sigma^2\}$ (in non-increasing order), there exists an eigen-gap $\lambda_r(\mathbf{\Sigma}) - \lambda_{r+1}(\mathbf{\Sigma}) = \lambda_r$, where $\lambda_r(\mathbf{\Sigma})$ denotes the r-th largest eigenvalue of $\mathbf{\Sigma}$. Therefore the first r leading eigenvalues of $\mathbf{\Sigma}$ can be regarded as "spikes" or signal eigenvalues, and the remaining eigenvalues σ^2 may be treated as "bulk" or noise eigenvalues. Here we assume that the eigenvector matrix $\mathbf{U}$ is jointly sparse, the formal definition of which is deferred later this section. Roughly speaking, joint sparsity refers to a significant amount of rows in $\mathbf{U}$ being zero, which allows for feature selection and brings easy interpretation in many applications. For example, in the analysis of face images, a classical method to extract common features among different facial characteristics, expressions, illumination conditions, etc., is to obtain the eigenvectors of these face data, referred to as eigenfaces. Each coordinate of these eigenvectors corresponds to a specific pixel in the image. Nonetheless, the number of pixels (features) is typically much larger than the number of images (samples), and it is often desirable to gain insights of the face information via a relatively small number of pixels, referred to as key pixels. By introducing joint sparsity to these eigenvectors, one is able to conveniently model key pixels among multiple face images corresponding to non-zero rows of eigenvectors. A concrete real data example is provided in Section 2.5.2 of Chapter 2 of the dissertation.

In the spiked covariance matrix model (1.1), we focus on the case where the leading r eigenvectors of $\mathbf{\Sigma}$ (the columns of $\mathbf{U}$) are jointly sparse [15, 27]. Formally, the row support of $\mathbf{U}$ is defined as

$$\mathrm{supp}(\mathbf{U}) = \left\{ j \in [p] : \mathbf{U}_{j*}^{\mathrm{T}} \neq \mathbf{0}_r \right\},$$

and $\mathbf{U}$ is said to be jointly s-sparse, if $|\mathrm{supp}(\mathbf{U})| \leq s$. Heuristically, this assumption asserts that the signal comes from at most s features among all p features. Geometrically, joint sparsity has the interpretation that at most s coordinates of $\mathbf{y}_i$ generate the subspace $\mathrm{Span}\{\mathbf{U}_{*1}, \ldots, \mathbf{U}_{*r}\}$ [27]. We note that $s \geq r$ due to the orthonormal constraint on the columns of $\mathbf{U}$.

Throughout we shall write $\mathbf{\Sigma}_0 = \mathbf{U}_0 \mathbf{\Lambda}_0 \mathbf{U}_0^{\mathrm{T}} + \sigma_0 \mathbf{I}_p$ to be the true covariance matrix that generates the data $\mathbf{Y} = [\mathbf{y}_1, \ldots, \mathbf{y}_n]^{\mathrm{T}}$ from the p-dimensional multivariate Gaussian distribution $\mathrm{N}_p(\mathbf{0}_p, \mathbf{\Sigma}_0)$, where $\mathbf{\Lambda}_0 = \mathrm{diag}(\lambda_{01}, \cdots, \lambda_{0r})$. The parameter space of interest for $\mathbf{\Sigma}$ is given by

$$\Theta(p, r, s) = \left\{ \mathbf{\Sigma} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^{\mathrm{T}} + \sigma^2 \mathbf{I}_p : \mathbf{U} \in \mathbb{O}(p, r), |\mathrm{supp}(\mathbf{U})| \leq s, \lambda_1 \geq \ldots \geq \lambda_r > 0 \right\}.$$

In the sparse spiked covariance matrix model, two inference tasks of interest are the estimation of the entire covariance matrix $\mathbf{\Sigma}$ and the estimation of the principal subspace $\mathrm{Span}\{\mathbf{U}_{*1}, \ldots, \mathbf{U}_{*r}\}$. The following minimax rate of convergence for $\mathbf{\Sigma}$ under the operator norm loss [15] serves as a benchmark for measuring the performance of any estimation procedure for $\mathbf{\Sigma}$.

Theorem 1 (15) *Let $1 \leq r \leq s \leq p$. Suppose that $(s \log p)/n \to 0$ and $\lambda_{01} \geq \lambda_{0r} > 0$ are bounded away from 0 and ∞. Then the minimax rate of convergence for estimating*

$\boldsymbol{\Sigma} \in \Theta(p, r, s)$ *is*

$$\inf_{\widehat{\boldsymbol{\Sigma}}} \sup_{\boldsymbol{\Sigma}_0 \in \Theta(p,r,s)} \mathbb{E}_{\boldsymbol{\Sigma}_0} \|\widehat{\boldsymbol{\Sigma}} - \boldsymbol{\Sigma}_0\|_2^2 \asymp \frac{s \log p}{n}. \tag{1.2}$$

Estimation of the principal subspace $\mathrm{Span}\{\mathbf{U}_{*1}, \ldots, \mathbf{U}_{*r}\}$ is less straightforward due to the fact that $\mathrm{Span}\{\mathbf{U}_{*1}, \ldots, \mathbf{U}_{*r}\}$ may not uniquely determine the eigenvector matrix $\mathbf{U}$. In particular, when there exist replicates among the eigenvalues $\{\lambda_1 + \sigma^2, \ldots, \lambda_r + \sigma^2\}$ (*i.e.*, $\lambda_k = \lambda_{k+1}$ for some $k \in [r-1]$), the corresponding eigenvectors $[\mathbf{U}_{*k}, \mathbf{U}_{*(k+1)}]$ can only be identified up to an orthogonal transformation. One solution is to focus on the Frobenius norm loss [28, 27] or the operator norm loss [15] of the corresponding projection matrix $\mathbf{U}\mathbf{U}^{\mathrm{T}}$, which is uniquely determined by $\mathrm{Span}\{\mathbf{U}_{*1}, \ldots, \mathbf{U}_{*r}\}$ and vice versa. The corresponding minimax rate of convergence for $\mathbf{U}\mathbf{U}^{\mathrm{T}}$ with respect to the *projection operator norm loss* $\|\widehat{\mathbf{U}}\widehat{\mathbf{U}}^{\mathrm{T}} - \mathbf{U}_0\mathbf{U}_0^{\mathrm{T}}\|_2$ is given by [15]:

$$\inf_{\widehat{\mathbf{U}}} \sup_{\boldsymbol{\Sigma}_0 \in \Theta(p,r,s)} \mathbb{E}_{\boldsymbol{\Sigma}_0} \|\widehat{\mathbf{U}}\widehat{\mathbf{U}}^{\mathrm{T}} - \mathbf{U}_0\mathbf{U}_0^{\mathrm{T}}\|_2^2 \asymp \frac{s \log p}{n}. \tag{1.3}$$

In Chapter 2, we will develop a Bayesian method that results in minimax-optimal estimators for the covariance matrix $\boldsymbol{\Sigma}$ as well as the principal subspace $\mathrm{Span}\{\mathbf{U}_{*1}, \ldots, \mathbf{U}_{*r}\}$. We will also investigate the performance of the resulting estimator for $\mathrm{Span}\{\mathbf{U}_{*1}, \ldots, \mathbf{U}_{*r}\}$ with regard to a novel loss function, referred to as the *two-to-infinity norm loss function*, which will be stated formally in Chapter 2. We will also elaborate on a sparsity enforcing matrix shrinkage prior in the spiked covariance matrix model, and establish the corresponding theory in Chapter 2 as well.

1.4 Background on random dot product graphs

Another class of structured high-dimensional models that has been gaining popularity for the recent decade is network models. In this dissertation, we take the random dot product graph model as one of the representatives of popular statistical network models, and we begin approaching it by first introducing the concept of the more general latent position graph model [21]. Due to the high-dimensionality nature and the complex structure of graph data, classical statistical methods for graph inference typically begin with dimensionality reduction, *i.e.*, to find low-dimensional representations of the vertices using a collection of points in some Euclidean space. These points are typically referred to as the *latent positions* of the vertices. The formal definition of the latent positions graph model is given as follows: Each vertex i in the graph is assigned a Euclidean vector $\mathbf{x}_i \in \mathbb{R}^d$, and the occurrence of an edge linking vertices i and j is a Bernoulli random variable with the success probability $\kappa(\mathbf{x}_i, \mathbf{x}_j)$, where $\kappa : \mathbb{R}^d \times \mathbb{R}^d \to [0, 1]$ is a symmetric link function. Apparently, practitioners will never have access to the true values of the latent positions in a latent position graph model given the observed network structure, but can only rely on reasonable estimates for them. These estimates for the latent positions, which are referred to as *embeddings* in the literature, can be further applied for certain post-dimensionality-reduction inference tasks, such as classification, clustering, regression, or hypothesis testing, etc.

The random dot product graph model [17], which is the second focus of this dissertation, is a particular class of latent position graphs taking the link function to be the dot product of the latent positions: $\kappa(\mathbf{x}_i, \mathbf{x}_j) = \mathbf{x}_i^{\mathrm{T}}\mathbf{x}_j$. The random dot product graph is of particular interest due to the following three reasons. Firstly, the adjacency matrix of a random dot product graph can be viewed as the sum of a low-rank matrix and a mean-zero noise matrix, which facilitates the use of low-rank matrix factorization (spectral decomposition) techniques for statistical inference on random dot produce graphs. Secondly, random dot product graphs can approximate general latent position graphs with positive symmetric definite link functions when the dimension of the latent positions d grows with the number of vertices at a certain rate [29]. Thirdly, random dot product graphs are flexible enough to include a variety of popular low-rank random graphs, *e.g.*, the stochastic block model, the degree-corrected stochastic block model, mixed-membership stochastic block model, etc. The readers are referred to the survey paper [30] for a thorough review of the recent development of random dot product graphs.

We now provide the formal definition of random dot product graphs. Denote $\mathcal{X} = \{\mathbf{x} = [x_1, \ldots, x_d]^{\mathrm{T}} \in \mathbb{R}^d : x_1, \ldots, x_d > 0, \|\mathbf{x}\| < 1\}$ the space of latent positions, and $\mathcal{X}^n = \{\mathbf{X} = [\mathbf{x}_1, \ldots, \mathbf{x}_n]^{\mathrm{T}} \in \mathbb{R}^{n\times d} : \mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathcal{X}\}$. Given an $n \times d$ matrix $\mathbf{X} = [\mathbf{x}_1, \ldots, \mathbf{x}_n]^{\mathrm{T}} \in \mathcal{X}^n$, a symmetric and hollow (*i.e.*, the diagonal entries are zeros) random matrix $\mathbf{A} = [A_{ij}]_{n\times n} \in \{0, 1\}^{n\times n}$ is said to be the adjacency matrix of a random dot product graph on n vertices $[n] = \{1, 2, \ldots, n\}$ with latent position matrix $\mathbf{X}$, denoted by $\mathbf{A} \sim \mathrm{RDPG}(\mathbf{X})$, if $A_{ij} \sim \mathrm{Bernoulli}(\mathbf{x}_i^{\mathrm{T}}\mathbf{x}_j)$ independently, $1 \leq i < j \leq n$. Namely, the distribution of $\mathbf{A}$ can be written as

$$p_{\mathbf{X}}(\mathbf{A}) = \prod_{i<j}(\mathbf{x}_i^{\mathrm{T}}\mathbf{x}_j)^{A_{ij}}(1 - \mathbf{x}_i^{\mathrm{T}}\mathbf{x}_j)^{1-A_{ij}}.$$

Example (Positive semidefinite stochastic block model) *The most popular example of the random dot product graph model is the stochastic block model with a positive semidefinite block probability matrix. Formally, given K with $K/n \to 0$, a symmetric random adjacency matrix $\mathbf{A} = [A_{ij}]_{n\times n}$ is drawn from a K-block stochastic block model with a symmetric block probability matrix $\mathbf{B} = [B_{kl}]_{K\times K} \in (0, 1)^{K\times K}$ and a block assignment function $\tau : [n] \to [K]$, denoted by $\mathbf{A} \sim \mathrm{SBM}(\mathbf{B}, \tau)$, if the random variables $A_{ij} \sim \mathrm{Bernoulli}(B_{\tau(i)\tau(j)})$ independently for $1 \leq i \leq j \leq n$. Namely, vertices in the same block have the same connecting probability. When $\mathbf{B}$ is positive semidefinite with rank d, we refer to the model as a positive semidefinite stochastic block model, and there exists a matrix $\mathbf{L} \in \mathbb{R}^{K\times d}$ such that $\mathbf{B} = \mathbf{L}\mathbf{L}^{\mathrm{T}}$. By converting the block assignment function τ into an $n \times K$ matrix $\mathbf{Z} = [\mathbb{1}\{\tau(i) = k\}]_{i\in[n],k\in[K]}$, we obtain $\mathbb{E}_{\mathbf{X}}(\mathbf{A}) = (\mathbf{Z}\mathbf{L})(\mathbf{Z}\mathbf{L})^{\mathrm{T}}$, and therefore, $\mathrm{SBM}(\mathbf{B}, \tau)$ coincides with $\mathrm{RDPG}(\mathbf{X})$ through the reparametrization $\mathbf{X} = \mathbf{Z}\mathbf{L}$. The positive semidefinite stochastic block model will be revisited in Section 3.4.*

Remark 1 *Not all stochastic block models can be represented by the random dot product* *graph model. Consider the following example:* $\mathbf{A} \sim \mathrm{SBM}(\mathbf{B}, \tau)$ *with* $\tau(1) = 1, \tau(2) = \ldots = \tau(n) = 2$*, where* $\mathbf{B} = [b_{k\ell}]_{2\times 2}$ *is indefinite, indicating that there exists some* $\mathbf{u} = [u_1, u_2]^{\mathrm{T}} \in \mathbb{R}^2$ *such that* $\mathbf{u}^{\mathrm{T}}\mathbf{B}\mathbf{u} < 0$*. Take* $\mathbf{v} = [u_1, u_2/(n-1), \ldots, u_2/(n-1)]^{\mathrm{T}} \in \mathbb{R}^n$*, and denote* $\mathbf{Z} = [\mathbb{1}\{\tau(i) = k\}]_{i\in[n],k\in[K]}$*. It follows that* $\mathbf{Z}^{\mathrm{T}}\mathbf{v} = \mathbf{u}$*, and hence,* $\mathbf{v}^{\mathrm{T}}\mathbb{E}(\mathbf{A})\mathbf{v} = (\mathbf{Z}^{\mathrm{T}}\mathbf{v})^{\mathrm{T}}\mathbf{B}(\mathbf{Z}^{\mathrm{T}}\mathbf{v}) = \mathbf{u}^{\mathrm{T}}\mathbf{B}\mathbf{u} < 0$*. Since* $\mathbb{E}(\mathbf{A})$ *is not positive semidefinite,* $\mathrm{SBM}(\mathbf{B}, \tau)$ *cannot be represented by* $\mathrm{RDPG}(\mathbf{X})$ *for some* $\mathbf{X} \in \mathbb{R}^{n\times 2}$*.*

Example (Hardy-Weinberg curve example) *We provide an example of the random dot product graph model that is very different from the stochastic block model in flavor, referred to as one instance of the latent structure random graph model introduced in [31]. Let* $d = 3$ *and* $C : (0, 1) \to \mathcal{X}^3$ *be the Hardy-Weinberg curve defined by* $C(t) = [t^2, 1 - 2t + t^2, 2t - 2t^2]^{\mathrm{T}} \in \mathbb{R}^3$*. Let* $(t_i)_{i=1}^n$ *be distinct points taking values in* $(0, 1)$*, and* $\mathbf{x}_i = C(t_i)$ *for all* $i \in [n]$*. Define the latent position matrix* $\mathbf{X}$ *by* $\mathbf{X} = [\mathbf{x}_1, \ldots, \mathbf{x}_n]^{\mathrm{T}} \in \mathbb{R}^{n\times 3}$*, and let* $\mathbf{A} \sim \mathrm{RDPG}(\mathbf{X})$*. The Hardy-Weinberg curve example was originally presented in [31] and will be revisited in Section 3.7.3.*

Remark 2 (Intrinsic non-identifiability) *We remark that the latent position matrix* $\mathbf{X}$ *cannot be uniquely determined by the distribution* $\mathbf{A} \sim \mathrm{RDPG}(\mathbf{X})$, i.e., $\mathbf{X}$ *is not identifiable. In fact, for any orthogonal matrix* $\mathbf{W} \in \mathbb{R}^{d\times d}$*, the two distributions* $\mathrm{RDPG}(\mathbf{X})$ *and* $\mathrm{RDPG}(\mathbf{X}\mathbf{W})$ *are identical, since for any* $i, j \in [n]$*,* $\mathbf{x}_i^{\mathrm{T}}\mathbf{x}_j = (\mathbf{W}\mathbf{x}_i)^{\mathrm{T}}(\mathbf{W}\mathbf{x}_j)$*. In addition, any* d*-dimensional random dot product graph model can be embedded into a* d'*-dimensional random dot product graph model for any* $d' > d$*, in the sense that there exists a* d'*-dimensional latent position matrix* $\mathbf{X}' \in \mathbb{R}^{n\times d'}$*, such that the two distributions* $\mathrm{RDPG}(\mathbf{X})$ *and* $\mathrm{RDPG}(\mathbf{X}')$ *are identical. The latter source of non-identifiability, however, can be eliminated by requiring the columns of* $\mathbf{X}$ *to be linearly independent.*

Since the latent position matrix $\mathbf{X}$ can only be identified up to an orthogonal transformation, one needs to properly rotate any embedding estimator $\widehat{\mathbf{X}}$ to align with the underlying true $\mathbf{X}$. The alignment matrix can be found by the solution to the orthogonal Procrustes problem $\mathbf{W}^* = \arg\inf_{\mathbf{W}} \|\widehat{\mathbf{X}}\mathbf{W} - \mathbf{X}\|_{\mathrm{F}}$, where the infimum ranges over the set of all orthogonal matrices in $\mathbb{R}^{d\times d}$ [30]. In particular, $\mathbf{W}^*$ has a closed-form expression. Consequently, one candidate loss function of interest has the form

$$L(\widehat{\mathbf{X}}, \mathbf{X}) = \frac{1}{n} \inf_{\mathbf{W}\in\mathbb{O}(d)} \|\widehat{\mathbf{X}} - \mathbf{X}\mathbf{W}\|_{\mathrm{F}}^2 = \inf_{\mathbf{W}\in\mathbb{O}(d)} \frac{1}{n} \sum_{i=1}^{n} \|\widehat{\mathbf{x}}_i - \mathbf{W}^{\mathrm{T}}\mathbf{x}_i\|_2^2,$$

where $\widehat{\mathbf{X}} = [\widehat{\mathbf{x}}_1, \ldots, \widehat{\mathbf{x}}_n]^{\mathrm{T}} \in \mathbb{R}^{n\times d}$. This loss function can also be interpreted as the average error of the embeddings $\widehat{\mathbf{x}}_1, \ldots, \widehat{\mathbf{x}}_n$ of all n vertices after the appropriate orthogonal alignment.

The adjacency matrix $\mathbf{A}$ can be viewed as the sum of a low-rank signal matrix $\mathbf{X}\mathbf{X}^{\mathrm{T}}$ and a noise matrix $\mathbf{E} = (e_{ij})_{n\times n}$, the elements of which are centered Bernoulli random variables $e_{ij} \sim \mathrm{Bernoulli}(\mathbf{x}_i^{\mathrm{T}}\mathbf{x}_j) - \mathbf{x}_i^{\mathrm{T}}\mathbf{x}_j$ that are independent for $1 \le i \le j \le n$. The authors of [32] argued for finding the embeddings using the adjacency matrix $\mathbf{A}$ directly by solving the least-squares problem

$$\widehat{\mathbf{X}} = \operatorname*{arg\,min}_{\mathbf{X}\in\mathbb{R}^{n\times d}} \|\mathbf{A} - \mathbf{X}\mathbf{X}^{\mathrm{T}}\|_{\mathrm{F}}^2.$$

The resulting estimator $\widehat{\mathbf{X}}$ is referred to as the *adjacency spectral embedding* (ASE) of $\mathbf{A}$ and is denoted by $\widehat{\mathbf{X}}_{\mathrm{ASE}}$. Theoretical properties of the adjacency spectral embedding have been relatively well-developed [33, 34, 19]. Specifically, the asymptotic characterization of the behavior of the ASE as an estimator for the latent position matrix has been established, including the consistency [32] and the limit of the sum of squared errors [35] as the number of vertices goes to infinity. Furthermore, for each individual vertex, the authors of [36] proved that the distribution of the corresponding row of the adjacency spectral embedding converges to a mean-zero multivariate normal distribution after proper scaling and centering, up to an orthogonal transformation, as the number of vertices goes to infinity. These theoretical studies of the spectral-based methods lay a solid foundation for the development of a broad range of subsequent inference tasks, including clustering [33, 37, 38], classification [32, 29], testing between graphs [35, 39], parameter estimation in latent structure random graphs [31], etc. The readers are referred to the survey paper [30] for a systematic review of the development of the theory for the ASE.

The following consistency result established in [32] features one of the first theoretical findings of the ASE:

Theorem 2 (32) *Suppose* $\mathbf{A} \sim \mathrm{RDPG}(\mathbf{X})$ *for some* $\mathbf{X} \in \mathbb{R}^{n\times d}$ *and* $(1/n)\mathbf{X}^{\mathrm{T}}\mathbf{X} \to \mathbf{\Delta}$ *for some positive definite* $\mathbf{\Delta} \in \mathbb{R}^{d\times d}$ *with distinct eigenvalues* $\lambda_1(\mathbf{\Delta}) > \ldots > \lambda_d(\mathbf{\Delta}) > 0$ *as* $n \to \infty$*. Assume that there exists* $\delta > 0$ *such that* $\min_{j\neq k} |\lambda_j(\mathbf{\Delta}) - \lambda_k(\mathbf{\Delta})| > 2\delta$ *and* $\lambda_d(\mathbf{\Delta}) > 2\delta$*. Then with probability greater than* $1 - 2(d^2+1)/n^2$*,*

$$\frac{1}{n}\inf_{\mathbf{W}\in\mathbb{O}(d)} \|\widehat{\mathbf{X}}_{\mathrm{ASE}} - \mathbf{X}\mathbf{W}\|_{\mathrm{F}}^2 \leq \frac{12d^2\log n}{\delta^3 n}. \tag{1.4}$$

Theorem 2 not only implies that the ASE is (first-order) consistent for $\mathbf{X}$ after an orthogonal alignment of $\widehat{\mathbf{X}}_{\mathrm{ASE}}$ towards $\mathbf{X}$ but also yields a convergence rate of the ASE with regard to the loss function L

$$L(\widehat{\mathbf{X}}_{\mathrm{ASE}}, \mathbf{X}) = o_{\mathbb{P}}\left(\frac{M_n \log n}{n}\right)$$

for arbitrary $M_n \to \infty$, where $(M_n)_{n=1}^{\infty}$ should be interpreted as a sequence converging to ∞ arbitrarily slowly. Nevertheless, as will be seen in Chapters 3 and 4, this rate is sub-optimal. We will also develop the minimax risk for estimating the latent position matrix $\mathbf{X}$ with respect to the loss $L(\cdot,\cdot)$ in Chapter 3, as well as constructing a fully Bayesian approach that results in a minimax/globally optimal point estimator. The distinct eigenvalues condition will also be relaxed in Chapters 3 and 4.

Chapter 2

Bayesian estimation of sparse spiked covariance matrices

2.1 Motivation and overview

The literature on sparse spiked covariance matrix estimation in high-dimensions from a frequentist perspective is relatively mature. In [11], it is shown that the classical principal component analysis can fail when $p \gg n$. In [28] and [27], the minimax estimation of the principal subspace (*i.e.*, the linear subspace spanned by the eigenvector matrix $\mathbf{U}$) with respect to the projection Frobenius norm loss under various sparsity structures on $\mathbf{U}$ is considered, and the authors of [15] provide minimax estimation procedures of the principal subspace with respect to the projection operator norm loss under the joint sparsity assumption.

In contrast, there is comparatively limited literature on Bayesian estimation of sparse spiked covariance matrices with theoretical guarantees. To the best of our knowledge, [40] and [41] are the only two papers in the literature addressing posterior contraction rates for Bayesian estimation of sparse spiked covariance matrix models. In particular, the authors of [41] discuss the posterior contraction behavior of the covariance matrix $\mathbf{\Sigma}$ with respect to the operator norm loss under the Dirichlet-Laplace shrinkage prior [42], but the contraction rates are sub-optimal when the number of spikes r grows with the sample size. In [40], the authors propose a carefully designed prior on $\mathbf{U}$ that yields rate-optimal posterior contraction of the principal subspace with respect to the projection Frobenius norm loss, but the tractability of computing the full posterior distribution is lost, except for the posterior mean as a point estimator. Neither the authors of [40] nor the authors of [41] discuss the posterior contraction behavior for sparse spiked covariance matrix models when the eigenvector matrix $\mathbf{U}$ exhibits joint sparsity.

We study a hierarchical Bayesian model for the spiked covariance matrices in this chapter of the dissertation. We quantify how well the proposed methodology estimates the entire covariance matrix $\boldsymbol{\Sigma}$ and the principal subspace $\mathrm{Span}\{\mathbf{U}_{*1}, \cdots, \mathbf{U}_{*r}\}$ in the high-dimensional and jointly sparse setup. This chapter of the dissertation features the following major contributions:

- We propose a matrix spike-and-slab LASSO prior to enforce the joint sparsity occurring in the eigenvector matrix $\mathbf{U}$ of the spiked covariance matrix. The matrix spike-and-slab LASSO prior is a novel continuous shrinkage prior that generalizes the classical spike-and-slab LASSO prior for vectors in [43] and [44] to jointly sparse rectangular matrices. In addition, we also present a collection of concentration and large deviation inequalities for the matrix spike-and-slab LASSO prior. These inequalities may be of independent interest as well. These inequalities not only serve as the main technical tools for deriving the posterior contraction results but also may be of independent interest as well.

- By enforcing the matrix spike-and-slab LASSO prior, we establish the rate-optimal posterior contraction for the entire covariance matrix $\boldsymbol{\Sigma}$ with respect to the operator norm loss as well as that for the principal subspace with respect to the projection operator norm loss.

- We also focus on the two-to-infinity norm loss, a novel loss function measuring the closeness between linear subspaces. As will be seen in Section 2.2, the two-to-infinity norm loss is able to detect element-wise perturbations of the eigenvector matrix $\mathbf{U}$ spanning the principal subspace. Under certain low-rank and bounded coherence conditions on $\mathbf{U}$, we obtain a tighter posterior contraction rate for the principal subspace with respect to the two-to-infinity norm loss than that with respect to the routinely used projection operator norm loss.

- Besides the contraction of the full posterior distribution, the Bayes procedure also leads to a point estimator for the principal subspace with a rate-optimal risk bound.

The rest of the chapter is organized as follows. We introduce the novel two-to-infinity norm loss function for uniform measurement of the principal subspace estimator in Section 2.2. Section 2.3 elaborates on the matrix spike-and-slab LASSO prior. Section 2.4 provides theoretical results, including the concentration and large deviation inequalities for the matrix spike-and-slab LASSO prior and the posterior contraction results. The numerical performance of the proposed methodology is presented in Section 2.5 through synthetic examples and the analysis of a real-world computer vision dataset. Further discussion is included in Section 2.6.

2.2 The two-to-infinity norm loss function

Recall that we introduce the projection operator norm loss function

$$\|\widehat{\mathbf{U}}\widehat{\mathbf{U}}^{\mathrm{T}} - \mathbf{U}_0\mathbf{U}_0^{\mathrm{T}}\|_2$$

for measuring the performance of a subspace estimator $\mathrm{Span}(\widehat{\mathbf{U}})$ in Section 1.3. The corresponding minimax rate is given by the authors of [15]:

$$\inf_{\widehat{\mathbf{U}}} \sup_{\mathbf{\Sigma}_0 \in \Theta(p,r,s)} \mathbb{E}_{\mathbf{\Sigma}_0} \|\widehat{\mathbf{U}}\widehat{\mathbf{U}}^{\mathrm{T}} - \mathbf{U}_0\mathbf{U}_0^{\mathrm{T}}\|_2^2 \asymp \frac{s\log p}{n}.$$

Though convenient, the direct estimation of the projection matrix $\mathbf{U}\mathbf{U}^{\mathrm{T}}$ does not provide insight into the element-wise errors of the principal eigenvectors $\{\mathbf{U}_{*1}, \dots, \mathbf{U}_{*r}\}$.

Motivated by a recent paper [45], the authors of which presents a collection of technical tools for the analysis of element-wise eigenvector perturbation bounds with respect to the two-to-infinity norm, we also focus on the following two-to-infinity norm loss

$$\|\widehat{\mathbf{U}} - \mathbf{U}_0\mathbf{W}_{\mathbf{U}}\|_{2\to\infty} \tag{2.1}$$

for estimating $\mathrm{Span}\{\mathbf{U}_{*1}, \ldots, \mathbf{U}_{*r}\}$ in addition to the projection operator norm loss, where $\mathbf{W}_{\mathbf{U}}$ is the orthogonal matrix given by

$$\mathbf{W}_{\mathbf{U}} = \arg\inf_{\mathbf{W}\in\mathbb{O}(r)} \|\widehat{\mathbf{U}} - \mathbf{U}_0\mathbf{W}\|_{\mathrm{F}}.$$

Here, $\mathbf{W}_{\mathbf{U}}$ corresponds to the orthogonal alignment of $\mathbf{U}_0$ so that $\widehat{\mathbf{U}}$ and $\mathbf{U}_0\mathbf{W}_{\mathbf{U}}$ are close in the Frobenius norm sense. As pointed out in [45], the use of $\mathbf{W}_{\mathbf{U}}$ as the orthogonal alignment matrix is preferred over the two-to-infinity alignment matrix

$$\mathbf{W}_{2\to\infty}^{\star} = \arg\inf_{\mathbf{W}\in\mathbb{O}(r)} \|\widehat{\mathbf{U}} - \mathbf{U}_0\mathbf{W}\|_{2\to\infty},$$

because $\mathbf{W}_{2\to\infty}$ is not analytically computable in general, whereas $\mathbf{W}_{\mathbf{U}}$ can be explicitly computed [46], facilitating the analysis: Let $\mathbf{U}_0^{\mathrm{T}}\widehat{\mathbf{U}}$ admit the singular value decomposition $\mathbf{U}_0^{\mathrm{T}}\widehat{\mathbf{U}} = \widetilde{\mathbf{U}}\widetilde{\mathbf{\Sigma}}\widetilde{\mathbf{V}}^{\mathrm{T}}$, then $\mathbf{W}_{\mathbf{U}} = \widetilde{\mathbf{U}}\widetilde{\mathbf{V}}^{\mathrm{T}}$.

The following lemma formalizes the connection between the projection operator norm loss and the two-to-infinity norm loss.

Lemma 1 *Let* $\mathbf{U}$ *and* $\mathbf{U}_0$ *be two orthonormal* r*-frames in* $\mathbb{R}^p$*, where* $2r < p$*. Then there exists an orthonormal* $2r$*-frame* $\mathbf{V}_{\mathbf{U}}$ *in* $\mathbb{R}^p$ *depending on* $\mathbf{U}$ *and* $\mathbf{U}_0$*, such that*

$$\|\mathbf{U} - \mathbf{U}_0\mathbf{W}_{\mathbf{U}}\|_{2\to\infty} \le \|\mathbf{V}_{\mathbf{U}}\|_{2\to\infty}\left(\|\mathbf{U}\mathbf{U}^{\mathrm{T}} - \mathbf{U}_0\mathbf{U}_0^{\mathrm{T}}\|_2 + \|\mathbf{U}\mathbf{U}^{\mathrm{T}} - \mathbf{U}_0\mathbf{U}_0^{\mathrm{T}}\|_2^2\right),$$

where $\mathbf{W}_{\mathbf{U}} = \arg\inf_{\mathbf{W}\in\mathbb{O}(r)} \|\mathbf{U} - \mathbf{U}_0\mathbf{W}\|_{\mathrm{F}}$ *is the Frobenius orthogonal alignment matrix.*

Proof. Let $\mathbf{U}_\perp$ and $\mathbf{U}_{0\perp} \in \mathbb{O}(p, p-r)$ be such that $[\mathbf{U}, \mathbf{U}_\perp]$ and $[\mathbf{U}_0, \mathbf{U}_{0\perp}] \in \mathbb{O}(p)$. By the CS decomposition (see, for example, Theorem 5.1 in [46]), there exists $\mathbf{U}_{11}, \mathbf{V}_{11} \in$

$\mathbb{O}(r)$ and $\mathbf{U}_{22}, \mathbf{V}_{22} \in \mathbb{O}(p-r)$, such that

$$\begin{bmatrix} \mathbf{U}_0^{\mathrm{T}}\mathbf{U} & \mathbf{U}_0^{\mathrm{T}}\mathbf{U}_\perp \\ \mathbf{U}_{0\perp}^{\mathrm{T}}\mathbf{U} & \mathbf{U}_{0\perp}^{\mathrm{T}}\mathbf{U}_\perp \end{bmatrix} = \begin{bmatrix} \mathbf{U}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{C} & -\mathbf{S} & \mathbf{0} \\ \mathbf{S} & \mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{(p-2r)} \end{bmatrix} \begin{bmatrix} \mathbf{V}_{11}^{\mathrm{T}} & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_{22}^{\mathrm{T}} \end{bmatrix}$$

where $\mathbf{C} = \mathrm{diag}(c_1, \ldots, c_r)$ and $\mathbf{S} = \mathrm{diag}(s_1, \ldots, s_r)$ are diagonal with non-negative entries, and $\mathbf{C}^2 + \mathbf{S}^2 = \mathbf{I}_r$. Write $\mathbf{U}_{22}$ into two blocks $\mathbf{U}_{22} = [\mathbf{U}_{221}, \mathbf{U}_{222}]$ with $\mathbf{U}_{221} \in \mathbb{O}(p-r, r)$. Take $\mathbf{Q} = [\mathbf{U}_0\mathbf{U}_{11}, \mathbf{U}_{0\perp}\mathbf{U}_{22}]$. Clearly, we have

$$\mathbf{Q}^{\mathrm{T}}\mathbf{U}_0\mathbf{U}_{11} = \begin{bmatrix} \mathbf{U}_{11}^{\mathrm{T}} & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_{22}^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \mathbf{U}_0^{\mathrm{T}} \\ \mathbf{U}_{0\perp}^{\mathrm{T}} \end{bmatrix} \mathbf{U}_0\mathbf{U}_{11} = \begin{bmatrix} \mathbf{I}_r \\ \mathbf{0}_r \\ \mathbf{0}_{p-2r} \end{bmatrix}$$

and

$$\mathbf{Q}^{\mathrm{T}}\mathbf{U}\mathbf{V}_{11} = \begin{bmatrix} \mathbf{U}_{11}^{\mathrm{T}} & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_{22}^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \mathbf{U}_0^{\mathrm{T}}\mathbf{U} \\ \mathbf{U}_{0\perp}^{\mathrm{T}}\mathbf{U} \end{bmatrix} \mathbf{V}_{11} = \begin{bmatrix} \mathbf{C} \\ \mathbf{S} \\ \mathbf{0}_{p-2r} \end{bmatrix}$$

Observe that $\|\mathbf{U}\mathbf{U}^{\mathrm{T}} - \mathbf{U}_0\mathbf{U}_0^{\mathrm{T}}\|_2 = \|\mathbf{S}\|_2$, and that $\mathbf{U}_0^{\mathrm{T}}\mathbf{U} = \mathbf{U}_{11}\mathbf{C}\mathbf{V}_{11}^{\mathrm{T}}$ is the singular value decomposition of $\mathbf{U}_0^{\mathrm{T}}\mathbf{U}$, implying that $\mathbf{W}_{\mathbf{U}} = \mathbf{U}_{11}\mathbf{V}_{11}^{\mathrm{T}}$. We proceed to compute

$$\begin{aligned}
\|\mathbf{U} - \mathbf{U}_0\mathbf{W}_{\mathbf{U}}\|_{2\to\infty} &\leq \|\mathbf{U} - \mathbf{U}_0\mathbf{U}_0^{\mathrm{T}}\mathbf{U}\|_{2\to\infty} + \|\mathbf{U}_0(\mathbf{U}_0^{\mathrm{T}}\mathbf{U} - \mathbf{W}_{\mathbf{U}})\|_{2\to\infty} \\
&= \left\|\mathbf{Q}\begin{bmatrix} \mathbf{S}^2 \\ -\mathbf{S}\mathbf{C} \\ \mathbf{0} \end{bmatrix}\right\|_{2\to\infty} + \left\|\mathbf{Q}\begin{bmatrix} \mathbf{C}(\mathbf{C} - \mathbf{I}_r) \\ -\mathbf{S}(\mathbf{C} - \mathbf{I}_r) \\ \mathbf{0} \end{bmatrix}\right\|_{2\to\infty} \\
&= \left\|\begin{bmatrix} \mathbf{U}_0 & \mathbf{U}_{0\perp} \end{bmatrix}\begin{bmatrix} \mathbf{U}_{11} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_{221} & \mathbf{U}_{222} \end{bmatrix}\begin{bmatrix} \mathbf{S}^2 \\ -\mathbf{S}\mathbf{C} \\ \mathbf{0} \end{bmatrix}\right\|_{2\to\infty} \\
&\quad + \left\|\begin{bmatrix} \mathbf{U}_0 & \mathbf{U}_{0\perp} \end{bmatrix}\begin{bmatrix} \mathbf{U}_{11} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_{221} & \mathbf{U}_{222} \end{bmatrix}\begin{bmatrix} \mathbf{C}(\mathbf{C} - \mathbf{I}_r) \\ -\mathbf{S}(\mathbf{C} - \mathbf{I}_r) \\ \mathbf{0} \end{bmatrix}\right\|_{2\to\infty} \\
&= \left\|\begin{bmatrix} \mathbf{U}_0 & \mathbf{U}_{0\perp} \end{bmatrix}\begin{bmatrix} \mathbf{U}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_{221} \end{bmatrix}\begin{bmatrix} \mathbf{S}^2 \\ -\mathbf{S}\mathbf{C} \end{bmatrix}\right\|_{2\to\infty} \\
&\quad + \left\|\begin{bmatrix} \mathbf{U}_0 & \mathbf{U}_{0\perp} \end{bmatrix}\begin{bmatrix} \mathbf{U}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_{221} \end{bmatrix}\begin{bmatrix} \mathbf{C}(\mathbf{C} - \mathbf{I}_r) \\ -\mathbf{S}(\mathbf{C} - \mathbf{I}_r) \end{bmatrix}\right\|_{2\to\infty} \\
&= \left\|\begin{bmatrix} \mathbf{U}_0\mathbf{U}_{11} & \mathbf{U}_{0\perp}\mathbf{U}_{221} \end{bmatrix}\begin{bmatrix} \mathbf{S}^2 \\ -\mathbf{S}\mathbf{C} \end{bmatrix}\right\|_{2\to\infty} \\
&\quad + \left\|\begin{bmatrix} \mathbf{U}_0\mathbf{U}_{11} & \mathbf{U}_{0\perp}\mathbf{U}_{221} \end{bmatrix}\begin{bmatrix} \mathbf{C}(\mathbf{C} - \mathbf{I}_r) \\ -\mathbf{S}(\mathbf{C} - \mathbf{I}_r) \end{bmatrix}\right\|_{2\to\infty}.
\end{aligned}$$

Denote $\mathbf{V}_{\mathbf{U}} = [\mathbf{U}_0\mathbf{U}_{11}, \mathbf{U}_{0\perp}\mathbf{U}_{221}]$. Clearly, $\mathbf{V}_{\mathbf{U}} \in \mathbb{O}(p, 2r)$:

$$\mathbf{V}_{\mathbf{U}}^{\mathrm{T}}\mathbf{V}_{\mathbf{U}} = \begin{bmatrix} \mathbf{U}_{11}^{\mathrm{T}} & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_{221}^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \mathbf{U}_0^{\mathrm{T}} \\ \mathbf{U}_{0\perp}^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \mathbf{U}_0 & \mathbf{U}_{0\perp} \end{bmatrix} \begin{bmatrix} \mathbf{U}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_{221} \end{bmatrix} = \mathbf{I}_{2r}.$$

Furthermore, by the previous derivation and the fact that $\|\mathbf{AB}\|_{2\to\infty} \leq \|\mathbf{A}\|_{2\to\infty}\|\mathbf{B}\|_2$, we have

$$
\begin{aligned}
\|\mathbf{U}-\mathbf{U}_0\mathbf{W}_{\mathbf{U}}\|_{2\to\infty} &\leq \|\mathbf{V}_{\mathbf{U}}\|_{2\to\infty}\left(\left\|\begin{bmatrix}\mathbf{S}^2\\ -\mathbf{SC}\end{bmatrix}\right\|_2 + \left\|\begin{bmatrix}\mathbf{C}(\mathbf{C}-\mathbf{I}_r)\\ -\mathbf{S}(\mathbf{C}-\mathbf{I}_r)\end{bmatrix}\right\|_2\right)\\
&= \|\mathbf{V}_{\mathbf{U}}\|_{2\to\infty}\left(\left\|\mathbf{S}^4+\mathbf{SC}^2\mathbf{S}\right\|_2^{1/2} + \left\|(\mathbf{C}-\mathbf{I}_r)^2\right\|_2^{1/2}\right)\\
&= \|\mathbf{V}_{\mathbf{U}}\|_{2\to\infty}\left(\|\mathbf{S}\|_2 + \|\mathbf{I}_r-\mathbf{C}\|_2\right)\\
&\leq \|\mathbf{V}_{\mathbf{U}}\|_{2\to\infty}\left(\|\mathbf{S}\|_2 + \left\|\mathbf{I}_r-\mathbf{C}^2\right\|_2\right)\\
&= \|\mathbf{V}_{\mathbf{U}}\|_{2\to\infty}\left(\left\|\mathbf{U}\mathbf{U}^{\mathrm{T}}-\mathbf{U}_0\mathbf{U}_0^{\mathrm{T}}\right\|_2 + \left\|\mathbf{U}\mathbf{U}^{\mathrm{T}}-\mathbf{U}_0\mathbf{U}_0^{\mathrm{T}}\right\|_2^2\right),
\end{aligned}
$$

and the proof is thus completed. $\square$

When the projection operator norm loss $\|\mathbf{U}\mathbf{U}^{\mathrm{T}}-\mathbf{U}_0\mathbf{U}_0^{\mathrm{T}}\|_2$ is much smaller than one, Lemma 1 states that the two-to-infinity norm loss can be upper bounded by the product of the projection operator norm loss and $\|\mathbf{V}_{\mathbf{U}}\|_{2\to\infty}$, where $\mathbf{V}_{\mathbf{U}} \in \mathbb{O}(p, 2r)$ is an orthonormal $2r$-frame in $\mathbb{R}^p$. In particular, under the sparse spiked covariance matrix models in high dimensions, the number of spikes r can be much smaller than the dimension p (*i.e.*, $\mathbf{V}_{\mathbf{U}}$ is a "tall and thin" rectangular matrix), and hence the factor $\|\mathbf{V}_{\mathbf{U}}\|_{2\to\infty}$ can be much smaller than $\max_{\mathbf{V}\in\mathbb{O}(p,2r)}\|\mathbf{V}\|_2 = 1$.

We provide the following motivating example for the insight that is unique to the two-to-infinity norm loss (2.1) in comparison with the projection operator norm loss for $\mathrm{Span}\{\mathbf{U}_{*1},\ldots,\mathbf{U}_{*r}\}$.

Example *Let $s \geq 4$ be even and $r = 1$. Suppose the truth $\mathbf{U}_0$ is given by*

$$
\mathbf{U}_0 = \Big[\underbrace{\tfrac{1}{\sqrt{s}}\ \ \ldots\ \ \tfrac{1}{\sqrt{s}}}_{s}\ \ \underbrace{0\ \ \ldots\ \ 0}_{p-s}\Big]^{\mathrm{T}},
$$

and consider the following two perturbations of $\mathbf{U}_0$:

$$
\widehat{\mathbf{U}}_1 = \Big[\underbrace{c(\epsilon)(\tfrac{1}{\sqrt{s}}+\epsilon)\ \ \ldots}_{s/2}\ \ \underbrace{c(\epsilon)(\tfrac{1}{\sqrt{s}}-\epsilon)\ \ \ldots}_{s/2}\ \ \underbrace{0\ \ \ldots\ \ 0}_{p-s}\Big]^{\mathrm{T}},
$$

$$
\widehat{\mathbf{U}}_2 = \Big[c(\delta)(\tfrac{1}{\sqrt{s}}+\delta)\ \ \underbrace{\tfrac{1}{\sqrt{s}}\ \ \ldots\ \ \tfrac{1}{\sqrt{s}}}_{s-2}\ \ c(\delta)(\tfrac{1}{\sqrt{s}}-\delta)\ \ \underbrace{0\ \ \ldots\ \ 0}_{p-s}\Big]^{\mathrm{T}},
$$

where $\epsilon > 0$ *is some sufficiently small perturbation,* $c(\epsilon)^2 = 1/(1+s\epsilon^2)$*, and* δ *is related to* ϵ *by*

$$c(\delta)^2 = \frac{1}{1+s\delta^2} = \frac{s}{2}\left(\frac{1}{\sqrt{1+s\epsilon^2}} - 1 + \frac{2}{s}\right).$$

The perturbed matrices $\widehat{\mathbf{U}}_1$ *and* $\widehat{\mathbf{U}}_2$ *are designed such that their projection operator norm losses are identical,* i.e., $\|\widehat{\mathbf{U}}_1\widehat{\mathbf{U}}_1^{\mathrm{T}} - \mathbf{U}_0\mathbf{U}_0^{\mathrm{T}}\|_2 = \|\widehat{\mathbf{U}}_2\widehat{\mathbf{U}}_2^{\mathrm{T}} - \mathbf{U}_0\mathbf{U}_0^{\mathrm{T}}\|_2$*. In contrast,* $\widehat{\mathbf{U}}_1$ *and* $\widehat{\mathbf{U}}_2$ *perturb* $\mathbf{U}_0$ *in different fashions: all* s *nonzero elements in* $\mathbf{U}_0$ *are perturbed in* $\widehat{\mathbf{U}}_1$*, whereas only two nonzero elements in* $\mathbf{U}_0$ *are perturbed in* $\widehat{\mathbf{U}}_2$*. We examine the*

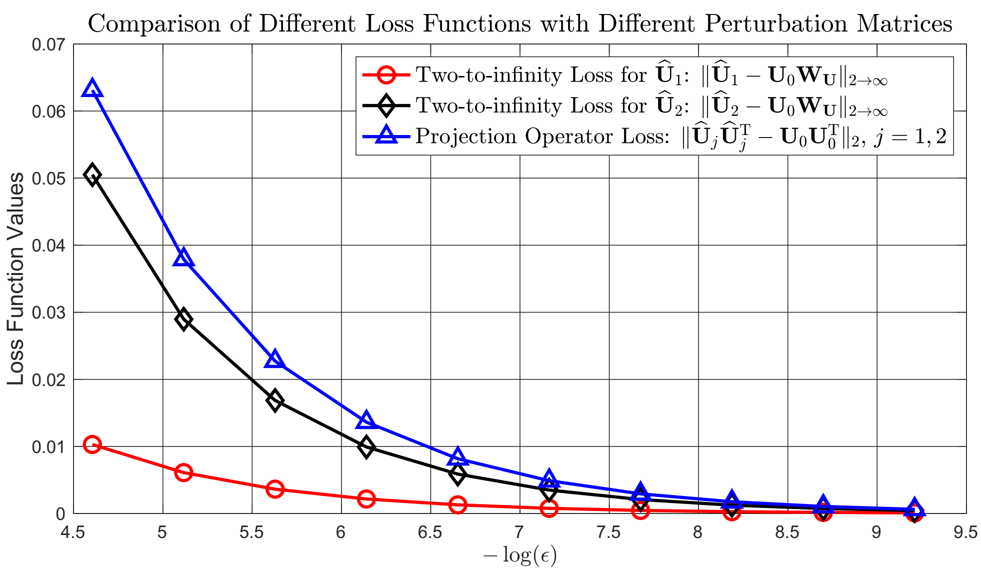


Figure 2-1. Motivating example: Comparison of different loss function values against different $-\log(\epsilon)$ values for two perturbed matrices $\widehat{\mathbf{U}}_1$ and $\widehat{\mathbf{U}}_2$.

two candidate losses $\|\widehat{\mathbf{U}}_j - \widehat{\mathbf{U}}_0\mathbf{W}_\mathbf{U}\|_{2\to\infty}$ *and* $\|\widehat{\mathbf{U}}_j\widehat{\mathbf{U}}_j^{\mathrm{T}} - \mathbf{U}_0\mathbf{U}_0^{\mathrm{T}}\|_2$ *for different values of* ϵ *and present them in Figure 2-1. It can clearly be seen that the two-to-infinity norm loss is smaller than the projection operator norm loss. Furthermore, the projection operator norm loss is unable to detect the difference between* $\widehat{\mathbf{U}}_1$ *and* $\widehat{\mathbf{U}}_2$*. In contrast, the two-to-infinity norm loss indicates that* $\widehat{\mathbf{U}}_2$ *has a larger element-wise deviation from* $\mathbf{U}_0$ *than* $\widehat{\mathbf{U}}_1$ *does. Thus the two-to-infinity norm loss is capable of detecting element-wise perturbations of the eigenvector compared to the projection operator norm* *loss for estimating* $\mathrm{Span}\{\mathbf{U}*1, \ldots, \mathbf{U}_{*r}\}$.

2.3 The matrix spike-and-slab LASSO prior

We begin approaching the desired matrix shrinkage prior by first illustrating the general Bayesian strategies in modeling sparsity occurring in high-dimensional statistics. Consider a simple yet canonical sparse normal mean problem. Suppose we observe independent normal data $y_i \sim \mathrm{N}(\beta_i, 1)$, $i = 1, \ldots, n$, with the goal of estimating the mean vector $\boldsymbol{\beta}_n = (\beta_i)_{i=1}^n$, which is assumed to be sparse in the sense that $\sum_{i=1}^n \mathbb{1}(|\beta_i| \neq 0) \leq s_n$ with the sparsity level $s_n = o(n)$ as $n \to \infty$. To model sparsity on $\boldsymbol{\beta}$, classical Bayesian methods impose the spike-and-slab prior of the following form on $\boldsymbol{\beta}$: For any $i \in [n]$,

$$\begin{aligned} \Pi(\mathrm{d}\beta_i \mid \lambda, \xi_i) &= (1 - \xi_i)\delta_0(\mathrm{d}\beta_i) + \xi_i \psi(\beta_i \mid \lambda)\mathrm{d}\beta_i, \\ (\xi_i \mid \theta) &\sim \mathrm{Bernoulli}(\theta), \end{aligned} \tag{2.2}$$

where ξ_i is the indicator that $\beta_i = 0$, $\theta \in (0, 1)$ represents the prior probability of β_i being non-zero, δ_0 is the point-mass at 0 (called the "spike" distribution), and $\psi(\cdot \mid \lambda)$ is the density of an absolutely continuous distribution (called the "slab" distribution) with respect to the Lebesgue measure on $\mathbb{R}$ governed by some hyperparameter λ. Theoretical justifications for the use of spike-and-slab prior (2.2) for sparse normal means and sparse Bayesian factor models have been established in [47] and [41], respectively. Therein, the spike-and-slab prior (2.2) involves point-mass mixtures, which can be daunting in terms of posterior simulations [41]. To address this issue, the authors of [43] designed the spike-and-slab LASSO prior as a continuous relaxation of (2.2):

$$\begin{aligned} \pi(\beta_i \mid \lambda_0, \lambda, \xi_i) &= (1 - \xi_i)\psi(\beta_i \mid \lambda_0) + \xi_i \psi(\beta_i \mid \lambda), \\ (\xi_i \mid \theta) &\sim \mathrm{Bernoulli}(\theta), \end{aligned} \tag{2.3}$$

where $\psi(\beta \mid \lambda) = (\lambda/2)\exp(-\lambda\beta)$ is the Laplace distribution with mean 0 and variance $2/\lambda^2$. When $\lambda_0 \gg \lambda$, the spike-and-slab LASSO prior (2.3) closely resembles the spike-and-slab prior (2.2). The continuity feature of the spike-and-slab LASSO prior (2.3), in contrast to the classical spike-and-slab prior (2.2), is highly desired in high-dimensional settings in terms of computation efficiency.

Motivated by the spike-and-slab LASSO prior, we develop a matrix spike-and-slab LASSO prior to model joint sparsity in sparse spiked covariance matrix models (1.1) with the covariance matrix $\mathbf{\Sigma} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^{\mathrm{T}} + \sigma^2\mathbf{I}_p$. The orthonormal constraint on the columns of $\mathbf{U}$ makes it challenging to incorporate prior distributions. Instead, we consider the following reparametrization of $\mathbf{\Sigma}$:

$$\mathbf{\Sigma} = \left(\mathbf{U}\mathbf{\Lambda}^{1/2}\mathbf{V}^{\mathrm{T}}\right)\left(\mathbf{U}\mathbf{\Lambda}^{1/2}\mathbf{V}^{\mathrm{T}}\right)^{\mathrm{T}} + \sigma^2\mathbf{I}_p = \mathbf{B}\mathbf{B}^{\mathrm{T}} + \sigma^2\mathbf{I}_p, \tag{2.4}$$

where $\mathbf{B} = \mathbf{U}\mathbf{\Lambda}^{1/2}\mathbf{V}^{\mathrm{T}} \in \mathbb{R}^{p\times r}$, and $\mathbf{V} \in \mathbb{O}(r)$ is an arbitrary orthogonal matrix in $\mathbb{R}^{r\times r}$. Clearly, in contrast to the orthonormal constraint on the columns of $\mathbf{U}$, there is no constraint on $\mathbf{B}$ except that $\mathrm{rank}(\mathbf{B}) = r$. Furthermore, $\mathbf{B}$ preserves the joint sparsity of $\mathbf{U}$: Specifically, for $|\mathrm{supp}(\mathbf{U})| = s \geq r$, there exists some permutation matrix $\mathbf{P} \in \mathbb{R}^{p\times p}$ and $\mathbf{U}^\star \in \mathbb{O}(s, r)$, such that

$$\mathbf{U} = \mathbf{P}\begin{bmatrix}\mathbf{U}^\star \\ \mathbf{0}_{(p-s)\times r}\end{bmatrix}.$$

It follows directly that

$$\mathbf{B} = \mathbf{U}\mathbf{\Lambda}^{1/2}\mathbf{V}^{\mathrm{T}} = \mathbf{P}\begin{bmatrix}\mathbf{U}^\star \\ \mathbf{0}_{(p-s)\times r}\end{bmatrix}\mathbf{\Lambda}^{1/2}\mathbf{V}^{\mathrm{T}} = \mathbf{P}\begin{bmatrix}\mathbf{U}^\star\mathbf{\Lambda}^{1/2}\mathbf{V}^{\mathrm{T}} \\ \mathbf{0}_{(p-s)\times r}\end{bmatrix},$$

implying that $|\mathrm{supp}(\mathbf{B})| \leq s$. Therefore, working with $\mathbf{B}$ allows us to circumvent the orthonormal constraint while maintaining the jointly sparse structure of $\mathbf{U}$. We propose the following matrix spike-and-slab LASSO prior on $\mathbf{B} = [b_{jk}]_{p\times r}$: Given hyperparameters $\lambda_0 > 0$ and $\theta \in (0, 1)$, for each $j \in [p]$, we independently assign the prior to $\mathbf{B}_{j*} = [b_{j1}, \ldots, b_{jr}]$ as follows:

$$\pi(b_{j1}, \ldots, b_{jr} \mid \lambda_0, \xi_j) = (1 - \xi_j)\prod_{k=1}^{r}\psi_r(b_{jk} \mid \lambda + \lambda_0) + \xi_j\prod_{k=1}^{r}\psi_1(b_{jk} \mid \lambda),$$

$$(\xi_j \mid \theta) \sim \text{Bernoulli}(\theta),$$

where $\boldsymbol{\xi} = [\xi_1, \ldots, \xi_p]^{\mathrm{T}} \in \{0,1\}^p$ are binary group assignment indicators, and $\psi_\alpha(x \mid \lambda)$ is the density function of the double Gamma distribution with shape parameter $1/\alpha$ and rate parameter λ:

$$\psi_\alpha(x \mid \lambda) = \frac{\lambda^{1/\alpha}}{2\Gamma(1/\alpha)}|x|^{1/\alpha-1}\exp(-\lambda|x|), \quad -\infty < x < \infty.$$

We further assign the following hyperpriors to λ_0 and θ:

$$\lambda_0 \sim \text{IGamma}(1/p^2, 1) \quad \text{and} \quad \theta \sim \text{Beta}\left(1, p^{1+\kappa}\right),$$

where $\text{IGamma}(a, b)$ is the inverse Gamma distribution with density

$$\pi(\lambda_0) \propto \lambda_0^{-a-1}\exp(-b/\lambda_0),$$

and $\kappa > 0$ is some fixed constant. We refer to the above hierarchical prior on $\mathbf{B}$ as the matrix spike-and-slab LASSO prior and denote $\mathbf{B} \sim \text{MSSL}_{p\times r}(\lambda, 1/p^2, p^{1+\kappa})$. The hyperparameter λ is fixed throughout. In the single-spike case ($r = 1$), we observe that $\psi_1(b_{jk} \mid \lambda) = (\lambda/2)\exp(-\lambda b_{jk})$ reduces to the density function of the Laplace distribution, and hence the matrix spike-and-slab LASSO prior coincides with the spike-and-slab LASSO prior [43].

Clearly, it can be seen that *a priori*, λ_0 is much larger than λ, so that $\xi_j = 0$ corresponds to a row $\mathbf{B}_{j*}$ that is close to $\mathbf{0}$, and $\xi_j = 1$ represents that the jth row is decently away from $\mathbf{0}$. It should be noted that unlike the spike-and-slab prior (2.2), the group indicator variable $\xi_j = 0$ or 1 corresponds to small or large values of $\mathbf{B}_{j*}$ rather than the exact sparsity of $\mathbf{B}_{j*}$. In addition, $\theta \sim \text{Beta}(1, p^{1+\kappa})$ indicates that the matrix spike-and-slab LASSO prior favors a large proportion of rows of $\mathbf{B}$ being close to $\mathbf{0}$. These features of the matrix spike-and-slab LASSO prior are in accordance with the joint sparsity assumption on $\mathbf{U}$. We complete the prior specification by letting $\sigma^2 \sim \text{IGamma}(a_\sigma, b_\sigma)$ for some $a_\sigma, b_\sigma > 0$ for the sake of conjugacy.

Lastly, we remark that the parametrization (2.4) of the spiked covariance matrix models (1.1) has another interpretation. The sampling model $\mathbf{y}_i \sim \mathrm{N}_p(\mathbf{0}_p, \boldsymbol{\Sigma})$ can be equivalently characterized in terms of the latent factor model

$$\mathbf{y}_i = \mathbf{B}\mathbf{z}_i + \boldsymbol{\varepsilon}_i, \quad \mathbf{z}_i \sim \mathrm{N}_r(\mathbf{0}_r, \mathbf{I}_r), \quad \boldsymbol{\varepsilon}_i \sim \mathrm{N}_p(\mathbf{0}_p, \sigma^2\mathbf{I}_p), \quad i = 1, \ldots, n, \tag{2.5}$$

where $\mathbf{z}_i$, $i = 1, \ldots, n$, are r-dimensional latent factors, $\mathbf{B}$ is a $p \times r$ factor loading matrix, and $\boldsymbol{\varepsilon}_i$, $i = 1, \ldots, n$ are homoscedastic noisy vectors. Since by our earlier discussion $\mathbf{B}$ is also sparse, this formulation is related to the sparse Bayesian factor models presented in [48] and [41], the differences being the joint sparsity of $\mathbf{B}$ and prior specifications on $\mathbf{B}$. In addition, the latent factor formulation (2.5) is convenient for posterior simulation through Markov chain Monte Carlo, as discussed in Section 3.1 of [48].

2.4 Theoretical properties

2.4.1 Properties of the matrix spike-and-slab LASSO prior

The theoretical properties of the classical spike-and-slab LASSO prior (2.3) have been partially explored in [43] and [44] in the context of sparse linear models and sparse normal means problems, respectively. It is not clear whether the properties of the spike-and-slab LASSO priors adapt to other statistical contexts, including sparse spiked covariance matrix models, high-dimensional multivariate regression [49], etc. In this subsection, we present a collection of theoretical properties of the matrix spike-and-slab LASSO prior that not only are useful for deriving posterior contraction under the spiked covariance matrix models, but also may be of independent interest for other statistical tasks, *e.g.*, sparse Bayesian linear regression with multivariate response [50].

Let $\mathbf{B} \in \mathbb{R}^{p\times r}$ be a $p\times r$ matrix, and let $\mathbf{B}_0 \in \mathbb{R}^{p\times r}$ be a jointly s-sparse $p\times r$ matrix with $r \leq s \leq p$, corresponding to the underlying truth. In the sparse spiked covariance matrix model, $\mathbf{B}$ represents the scaled eigenvector matrix $\mathbf{U}\mathbf{\Lambda}^{1/2}$ up to an orthonormal matrix in $\mathbb{O}(r)$, but for generality, we do not impose the statistical context in this subsection. A fundamental measure of goodness for various prior models with high dimensionality is the prior mass assignment on a small neighborhood around the true but unknown value of the parameter. This is referred to as the *prior concentration* in the literature of Bayes theory. Formally, we consider the prior probability of the non-centered ball $\{\|\mathbf{B} - \mathbf{B}_0\|_{\mathrm{F}} < \eta\}$ under the prior distribution for small values of η.

Lemma 2 *Suppose* $\mathbf{B} \sim \mathrm{MSSL}_{p\times r}(\lambda, 1/p^2, p^{1+\kappa})$ *for some fixed positive constants* λ *and* κ*, and* $\mathbf{B}_0 \in \mathbb{R}^{p\times r}$ *is jointly* s*-sparse, where* $1 \leq r \leq s \leq p/2$*. Then for small values of* $\eta \in (0, 1)$ *with* $\eta \geq 1/p^\gamma$ *for some* $\gamma > 0$*, it holds that*

$$\Pi\left(\|\mathbf{B} - \mathbf{B}_0\|_{\mathrm{F}} < \eta\right) \geq \exp\left[-C_1 \max\left\{\lambda^2 s\|\mathbf{B}_0\|_{2\to\infty}^2, sr\left|\log\frac{\lambda\eta}{\sqrt{sr}}\right|, s\log p\right\}\right]$$

for some absolute constant $C_1 > 0$*.*

Next, we formally characterize how the matrix spike-and-slab LASSO prior enforces joint sparsity on the rows of $\mathbf{B}$ using a probabilistic argument. Unlike the classical spike-and-slab prior (2.2), which allows the occurrence of exact zeros in the mean vector with positive probability, the spike-and-slab LASSO prior (2.3) (the matrix spike-and-slab LASSO prior) is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^n$ ($\mathbb{R}^{p\times r}$, respectively), and $|\mathrm{supp}(\mathbf{B})| = p$ with probability one. Rather than forcing rows of $\mathbf{B}$ to be exactly $\mathbf{0}$, the matrix spike-and-slab LASSO prior shrinks the rows of $\mathbf{B}$ toward $\mathbf{0}$. This behavior suggests the following generalization of the row support of a matrix $\mathbf{B}$: For $\delta > 0$ taken to be small, we define $\mathrm{supp}_\delta(\mathbf{B}) = \{j \in [p] : \|\mathbf{B}_{j*}\|_2 > \delta\}$. Namely, $\mathrm{supp}_\delta(\mathbf{B})$ consists of indices of the rows of $\mathbf{B}$ whose Euclidean norms are greater than δ. Intuitively, one should expect that under the matrix spike-and-slab LASSO prior, $|\mathrm{supp}_\delta(\mathbf{B})|$ should be small with a large probability. The following lemma formally confirms this intuition.

Lemma 3 *Suppose* $\mathbf{B} \sim \mathrm{MSSL}_{p\times r}(\lambda, 1/p^2, p^{1+\kappa})$ *for some fixed positive constants* λ

and $\kappa \leq 1$*,* $1 \leq r \leq p$*. Let* $\delta \in (0,1)$ *be a small number with* $\delta > 1/p^{\gamma}$ *for some* $\gamma > 0$*, and let* s *be an integer such that* $(s\log p)/p$ *is sufficiently small. Then for any* $\beta > 4\gamma\exp(1)$*, it holds that*

$$\Pi\left(|\mathrm{supp}_{\delta}(\mathbf{B})| > \beta s\right) \leq 2\exp\left\{-\min\left(\frac{\beta\kappa}{2}, \frac{\beta}{2\mathrm{e}} - 2\gamma\right)s\log p\right\}.$$

We conclude this section by providing a large deviation inequality for the matrix spike-and-slab LASSO prior.

Lemma 4 *Suppose* $\mathbf{B} \sim \mathrm{MSSL}_{p\times r}(\lambda, 1/p^2, p^{1+\kappa})$ *for some fixed positive* λ *and* $\kappa < 1$*, and* $\mathbf{B}_0 \in \mathbb{R}^{p\times r}$ *is jointly* s*-sparse, where* $r\log n \lesssim \log p$*, and* $(s\log p)/p$ *is sufficiently small. Let* $(\delta_n)_{n=1}^{\infty}$ *and* $(t_n)_{n=1}^{\infty}$ *be positive sequences such that* $1/p^{\gamma} \leq \delta_n \leq 1$ *and* $t_n/(sr) \to \infty$*. Then for sufficiently large* n *and for all* $\beta > 4\gamma\exp(1)$*, it holds that*

$$\begin{aligned}
&\Pi\left[\sum_{j=1}^{p}\|\mathbf{B}_{j*}\|_1\mathbb{1}\{j \in \mathrm{supp}_{\delta_n}(\mathbf{B}) \cup \mathrm{supp}(\mathbf{B}_0)\} \geq t_n\right]\\
&\quad\leq 2\exp\left[-C_2\min\left\{\left(\frac{t_n}{\beta sr}\right)^2, \left(\frac{t_n}{r}\right)^2, \frac{t_n}{r}\right\}\right] + 3\exp\left\{-\min\left(\frac{\beta\kappa}{2}, \frac{\beta}{2\mathrm{e}} - 2\gamma\right)s\log p\right\}
\end{aligned}$$

for some absolute constant $C_2 > 0$*.*

2.4.2 Posterior contraction results

We now present the posterior contraction rates for sparse spiked covariance matrix models under the matrix spike-and-slab LASSO prior with respect to various loss functions, which are the main results of this chapter. We point out that the posterior contraction rates presented in the following theorem are minimax-optimal as they coincide with (1.2) and (1.3).

Theorem 3 *Assume the data* $\mathbf{y}_1, \ldots, \mathbf{y}_n$ *are independently sampled from* $\mathrm{N}_p(\mathbf{0}_p, \mathbf{\Sigma}_0)$ *with* $\mathbf{\Sigma}_0 = \mathbf{U}_0\mathbf{\Lambda}_0\mathbf{U}_0^{\mathrm{T}} + \sigma_0^2\mathbf{I}_p$*,* $\mathbf{\Lambda}_0 = \mathrm{diag}(\lambda_{01}, \ldots, \lambda_{0r})$*,* $|\mathrm{supp}(\mathbf{U}_0)| \leq s$*, and* $1 \leq r \leq s \leq p$*. Suppose* $(s\log p)/n \to 0$*,* $p/n \to \infty$*, and* $r\log n \lesssim \log p$*. Let* $\mathbf{B} \sim \mathrm{MSSL}_{p\times r}(\lambda, 1/p^2, p^{1+\kappa})$ *for some positive* $\lambda > 0$ *and* $\kappa \leq 1$*, and* $\sigma^2 \sim \mathrm{IGamma}(a_{\sigma}, b_{\sigma})$

for some $a_\sigma, b_\sigma \geq 1$. Then there exists some constants $M_0 > 0$, R_0, and C_0 depending on σ_0 and $\mathbf{\Lambda}_0$, and hyperparameters, such that the following posterior contraction for $\mathbf{\Sigma} = \mathbf{B}\mathbf{B}^{\mathrm{T}} + \sigma^2\mathbf{I}_p$ holds for all $M \geq M_0$ when n is sufficiently large:

$$\mathbb{E}_0\left\{\Pi\left(\|\mathbf{\Sigma} - \mathbf{\Sigma}_0\|_2 > M\sqrt{\frac{s\log p}{n}} \;\middle|\; \mathbf{Y}_n\right)\right\} \leq R_0\exp(-C_0 s\log p). \tag{2.6}$$

For each $\mathbf{B}$, let $\mathbf{U}_{\mathbf{B}} \in \mathbb{O}(p, r)$ be the left-singular vector matrix of $\mathbf{B}$. Then the following posterior contraction for $\mathbf{U}_{\mathbf{B}}$ holds for all $M \geq M_0$:

$$\mathbb{E}_0\left\{\Pi\left(\|\mathbf{U}_{\mathbf{B}}\mathbf{U}_{\mathbf{B}}^{\mathrm{T}} - \mathbf{U}_0\mathbf{U}_0^{\mathrm{T}}\|_2 > \frac{2M}{\lambda_{0r}}\sqrt{\frac{s\log p}{n}} \;\middle|\; \mathbf{Y}_n\right)\right\} \leq R_0\exp(-C_0 s\log p). \tag{2.7}$$

Remark 3 *We briefly compare the posterior contraction rates obtained in Theorem 3 with some related results in the literature. In [41] the authors consider the posterior contraction with respect to the operator norm loss $\|\mathbf{\Sigma} - \mathbf{\Sigma}_0\|_2$ of the entire covariance matrix, while in [40], the authors consider the posterior contraction with respect to the projection Frobenius norm loss $\|\mathbf{U}\mathbf{U}^{\mathrm{T}} - \mathbf{U}_0\mathbf{U}_0^{\mathrm{T}}\|_{\mathrm{F}}$ for estimating $\mathrm{Span}\{\mathbf{U}_{*1}, \ldots, \mathbf{U}_{*r}\}$. In [41], the notion of sparsity is slightly different from the joint sparsity notion presented here, as they assume that under the latent factor model representation (2.5), the individual supports of columns of $\mathbf{B}$ are not necessarily the same. When $r = O(1)$, the assumption in [41] coincides with this chapter, and our rate $\epsilon_n = \sqrt{(s\log p)/n}$ is superior to the rate $\sqrt{(s\log p\log n)/n}$ obtained in [41] by a logarithmic factor. The assumptions in [40] are the same as those in [41], and in [40] the authors focus on designing a prior that yields rate-optimal posterior contraction with respect to the Frobenius norm loss of the projection matrices as well as adapting to the prior sparsity s and the rank r. Our result in equation (2.7), which focuses on the projection operator norm loss, serves as a complement to the rate-optimal posterior contraction for principal subspaces under the joint sparsity assumption in contrast to [40], in which the authors work on the projection Frobenius norm loss.*

To derive the posterior contraction rate for the principal subspace with respect to the two-to-infinity norm loss, we need the posterior contraction result for $\mathbf{\Sigma}$ with respect to the stronger matrix infinity norm. These two results are summarized in the following theorem.

Theorem 4 *Assume the conditions in Theorem 3 hold. Further assume that the eigenvector matrix* $\mathbf{U}_0$ *exhibits bounded coherence:* $\|\mathbf{U}_0\|_{2\to\infty} \leq C_\mu\sqrt{r/s}$ *for some constant* $C_\mu \geq 1$*, and the number of spikes* r *is sufficiently small in the sense that* $r^3/s = O(1)$*. Then there exists some constants* $M_{2\to\infty} > 0$ *depending on* σ_0 *and* $\mathbf{\Lambda}_0$*, and hyperparameters, such that the following posterior contraction for* $\mathbf{\Sigma} = \mathbf{B}\mathbf{B}^{\mathrm{T}} + \sigma^2\mathbf{I}_p$ *holds for all* $M \geq M_{2\to\infty}$ *when* n *is sufficiently large:*

$$\mathbb{E}_0\left\{\Pi\left(\|\mathbf{\Sigma} - \mathbf{\Sigma}_0\|_\infty > Mr\sqrt{\frac{s\log p}{n}}\;\middle|\;\mathbf{Y}_n\right)\right\} \leq R_0\exp(-C_0 s\log p), \tag{2.8}$$

For each $\mathbf{B}$*, let* $\mathbf{U}_{\mathbf{B}} \in \mathbb{O}(p, r)$ *be the left-singular vector matrix of* $\mathbf{B}$*. Then the following posterior contraction for* $\mathbf{U}_{\mathbf{B}}$ *holds for all* $M \geq M_0$*:*

$$\mathbb{E}_0\left[\Pi\left\{\|\mathbf{U}_{\mathbf{B}} - \mathbf{U}_0\mathbf{W}_{\mathbf{U}}\|_{2\to\infty} > M\left(\sqrt{\frac{r^3\log p}{n}} \vee \frac{s\log p}{n}\right)\right\}\right] \leq 2R_0\exp(-C_0 s\log p), \tag{2.9}$$

where $\mathbf{W}_{\mathbf{U}}$ *is the Frobenius orthogonal alignment matrix*

$$\mathbf{W}_{\mathbf{U}} = \underset{\mathbf{W}\in\mathbb{O}(r)}{\arg\inf}\|\mathbf{U}_{\mathbf{B}} - \mathbf{U}_0\mathbf{W}\|_{\mathrm{F}}.$$

Remark 4 *We also present some remarks concerning the posterior contraction with respect to the two-to-infinity norm loss* $\|\mathbf{U} - \mathbf{U}_0\mathbf{W}_{\mathbf{U}}\|_{2\to\infty}$*. In [45], the authors show that*

$$\|\mathbf{U} - \mathbf{U}_0\mathbf{W}_{\mathbf{U}}\|_{2\to\infty} \leq \|\mathbf{U} - \mathbf{U}_0\mathbf{W}_{\mathbf{U}}\|_2 \lesssim \|\mathbf{U}\mathbf{U}^{\mathrm{T}} - \mathbf{U}_0\mathbf{U}_0^{\mathrm{T}}\|_2,$$

meaning that $\|\mathbf{U} - \mathbf{U}_0\mathbf{W}_{\mathbf{U}}\|_{2\to\infty}$ *can be coarsely upper bounded by the projection operator norm loss* $\|\mathbf{U}\mathbf{U}^{\mathrm{T}} - \mathbf{U}_0\mathbf{U}_0^{\mathrm{T}}\|_2$*. This naive bound immediately yields*

$$\mathbb{E}_0\left\{\Pi\left(\|\mathbf{U}_{\mathbf{B}} - \mathbf{U}_0\mathbf{W}_{\mathbf{U}}\|_{2\to\infty} > M\sqrt{\frac{s\log p}{n}}\;\middle|\;\mathbf{Y}_n\right)\right\} \leq R_0\exp(-C_0 s\log p)$$

for some large M*, which is the same as (2.7). Our result (2.9) improves this rate by a factor of* $\{\sqrt{r^3/s} \vee \sqrt{(s\log p)/n}\}$*, resulting in a tighter posterior contraction rate with* *respect to the two-to-infinity norm loss. In particular, when $r \ll s$ (*i.e.*, $\mathbf{U}_0$ is a "tall and thin" rectangular matrix), the factor $\sqrt{r^3/s}$ can be much smaller than 1.*

The posterior contraction rate (2.7) also leads to the following risk bound for a point estimator of the principal subspace $\mathrm{Span}\{\mathbf{U}_{*1}, \ldots, \mathbf{U}_{*r}\}$ with regard to the projection operator norm loss:

Theorem 5 *Assume the conditions in Theorem 3 hold. Let*

$$\widehat{\mathbf{\Omega}} = \int \mathbf{U}_{\mathbf{B}}\mathbf{U}_{\mathbf{B}}^{\mathrm{T}}\Pi(\mathrm{d}\mathbf{B} \mid \mathbf{Y}_n)$$

be the posterior mean of the projection matrix $\mathbf{U}_{\mathbf{B}}\mathbf{U}_{\mathbf{B}}^{\mathrm{T}}$, and set $\widehat{\mathbf{U}} \in \mathbb{O}(p, r)$ be the orthonormal r-frame in $\mathbb{R}^p$ with columns being the first r eigenvectors corresponding to the first r largest eigenvalues of $\widehat{\mathbf{\Omega}}$. Then the following risk bound holds for $\widehat{\mathbf{U}}$ for sufficiently large n:

$$\mathbb{E}_0\left(\|\widehat{\mathbf{U}}\widehat{\mathbf{U}}^{\mathrm{T}} - \mathbf{U}_0\mathbf{U}_0^{\mathrm{T}}\|_2\right) \leq \left(\frac{4M_0}{\lambda_{0r}} + 4\sqrt{R_0}\right)\sqrt{\frac{s\log p}{n}}.$$

The setup so far is concerned with the case where r is known and fixed. When r is unknown, the authors of [28] provide a diagonal thresholding method for consistently estimating r. In such a setting, the posterior contraction in Theorem 3 reduces to the following weaker version:

Corollary 1 *Assume the data $\mathbf{y}_1, \ldots, \mathbf{y}_n$ are independently sampled from $\mathrm{N}_p(\mathbf{0}_p, \mathbf{\Sigma}_0)$ with $\mathbf{\Sigma}_0 = \mathbf{U}_0\mathbf{\Lambda}_0\mathbf{U}_0^{\mathrm{T}} + \sigma_0^2\mathbf{I}_p$, $\mathbf{\Lambda}_0 = \mathrm{diag}(\lambda_{01}, \ldots, \lambda_{0r})$, $|\mathrm{supp}(\mathbf{U}_0)| \leq s$, and $1 \leq r \leq s \leq p$. Suppose $(s\log p)/n \to 0$, $p/n \to \infty$, and $r\log n \lesssim \log p$, but r is unknown and instead is consistently estimated by $\hat{r}$ (*i.e.*, $\mathbb{P}_0(\hat{r} = r) \to 1$). Let $\mathbf{B} \sim \mathrm{MSSL}_{p\times\hat{r}}(\lambda, 1/p^2, p^{1+\kappa})$ for some positive $\lambda > 0$ and $\kappa \leq 1$, and $\sigma^2 \sim \mathrm{IGamma}(a_\sigma, b_\sigma)$ for some $a_\sigma, b_\sigma \geq 1$. Then there exists some large constant $M_0 > 0$, such that the following posterior contraction for $\mathbf{\Sigma}$ holds for all $M \geq M_0$:*

$$\lim_{n\to\infty}\mathbb{E}_0\left\{\Pi\left(\|\mathbf{\Sigma} - \mathbf{\Sigma}_0\|_2 > M\sqrt{\frac{s\log p}{n}}\,\middle|\,\mathbf{Y}_n\right)\right\} \to 0.$$

For each $\mathbf{B}$*, let* $\mathbf{U}_{\mathbf{B}} \in \mathbb{O}(p, \hat{r})$ *be the left-singular vector matrix of* $\mathbf{B}$*. Then the following posterior contraction for* $\mathbf{U}$ *holds for all* $M \geq M_0$*:*

$$\lim_{n\to\infty} \mathbb{E}_0 \left\{ \Pi \left(\|\mathbf{U}_{\mathbf{B}}\mathbf{U}_{\mathbf{B}}^{\mathrm{T}} - \mathbf{U}_0\mathbf{U}_0^{\mathrm{T}}\|_2 > \frac{2M}{\lambda_{0r}}\sqrt{\frac{s\log p}{n}} \,\middle|\, \mathbf{Y}_n \right) \right\} \to 0.$$

2.4.3 Proof sketch and auxiliary results

Now we sketch the proof of Theorem 3 along with some important auxiliary results. The proof strategy is based on a modification of the standard testing-and-prior-concentration approach, which was originally developed in [51] for proving convergence rates of posterior distributions, and later applied to a variety of statistical contexts. Specialized to the sparse spiked covariance matrix models, let us consider the posterior contraction for $\boldsymbol{\Sigma}$ with respect to the operator norm loss as an example. The posterior contraction for $\boldsymbol{\Sigma}$ with respect to the infinity norm loss can be proved in a similar fashion. Denote $\mathcal{U}_n = \{\boldsymbol{\Sigma} : \|\boldsymbol{\Sigma} - \boldsymbol{\Sigma}_0\|_2 \leq M\epsilon_n\}$, and write the posterior distribution as

$$\Pi(\mathcal{U}_n^c \mid \mathbf{Y}_n) = \frac{\int_{\mathcal{U}_n^c} \exp\{\ell_n(\boldsymbol{\Sigma}) - \ell_n(\boldsymbol{\Sigma}_0)\}\Pi(\mathrm{d}\boldsymbol{\Sigma})}{\int \exp\{\ell_n(\boldsymbol{\Sigma}) - \ell_n(\boldsymbol{\Sigma}_0)\}\Pi(\mathrm{d}\boldsymbol{\Sigma})} = \frac{N_n(\mathcal{U}_n)}{D_n}, \tag{2.10}$$

where $\ell_n(\boldsymbol{\Sigma})$ is the log-likelihood function of $\boldsymbol{\Sigma}$ given by

$$\ell_n(\boldsymbol{\Sigma}) = \sum_{i=1}^n \log p(\mathbf{y}_i \mid \boldsymbol{\Sigma}) = \sum_{i=1}^n \left\{ -\frac{1}{2}\log\det(2\pi\boldsymbol{\Sigma}) - \frac{1}{2}\mathbf{y}_i^{\mathrm{T}}\boldsymbol{\Sigma}^{-1}\mathbf{y}_i \right\}.$$

To provide a useful upper bound for $\mathbb{E}_0\{\Pi(\mathcal{U}_n^c \mid \mathbf{Y}_n)\}$ (*e.g.*, $\exp(-C_0 s\log p)$ appearing in Theorem 3), we modify the original testing-and-prior-concentration approach and require that the following three conditions hold:

1. **Prior concentration condition.** The prior distribution provides sufficient concentration around the true $\boldsymbol{\Sigma}_0$: There exists some constant $C_3 > 0$ such that

   $$\Pi(\|\boldsymbol{\Sigma} - \boldsymbol{\Sigma}_0\|_{\mathrm{F}}^2 \leq sr/n) \geq \exp(-C_3 s\log p)$$

   for sufficient large n.

2. **Existence of Tests.** There exists a sequence of subsets $(\mathcal{F}_n)_{n=1}^{\infty}$ of $\Theta(p, r, s)$, such that $\Pi(\mathbf{\Sigma} \in \mathcal{F}_n^c) \le \exp(-C_4 s \log p)$ for some sufficiently large constant $C_4 > 0$, and a sequence of test functions $(\phi_n)_{n=1}^{\infty}$, such that

$$\mathbb{E}_0(\phi_n) \lesssim \exp\left(-C_{41}\sqrt{M}n\epsilon_n^2\right),$$
$$\sup_{\mathbf{\Sigma}\in\mathcal{U}_n^c\cap\mathcal{F}_n} \mathbb{E}_{\mathbf{\Sigma}}(1-\phi_n) \lesssim \exp(-C_{42}Mn\epsilon_n^2)$$

for some constants $C_{41}, C_{42} > 0$.

The prior concentration condition can be verified by invoking Lemma 2. This condition is useful, as it guarantees that the denominator D_n appearing in the right-hand side of (2.10) can be lower bounded with high probability. The following lemma formalizes this result.

Lemma 5 *Let $\mathcal{K}_n(\eta) = \{\|\mathbf{\Sigma} - \mathbf{\Sigma}_0\|_{\mathrm{F}} \le \eta\}$ and $\eta < \sigma_0^2/2$. Then there exists some event $\mathcal{A}_n$ such that*

$$\mathcal{A}_n \subset \left\{D_n \ge \Pi_n\{\mathbf{\Sigma} \in \mathcal{K}_n(\eta)\} \exp\left[-\left\{\frac{C_3 \log \rho}{2(\lambda_{0r} + \sigma_0^2)} + 1\right\} n\eta^2\right]\right\}$$

for some absolute constant $C_3 > 0$, and

$$\mathbb{P}_0(\mathcal{A}_n^c) \le 2\exp\left\{-\tilde{C}_3 \min\left(\frac{n\eta^2}{\|\mathbf{\Sigma}_0^{-1}\|_2^2}, n\eta^2\right)\right\},$$

where $\rho = 2(\lambda_{01} + \sigma_0^2)/(\lambda_{0r} + \sigma_0^2)$ depends on the spectra of $\mathbf{\Sigma}$ only, and $\tilde{C}_3 > 0$ is an absolute constant.

Verifying the existence of tests is slightly more involved. It relies on Lemma 3, Lemma 4, and the following auxiliary lemma.

Lemma 6 *Assume the data $\mathbf{y}_1, \ldots, \mathbf{y}_n$ follow $\mathrm{N}_p(\mathbf{0}_p, \mathbf{\Sigma})$, $1 \le r \le p$. Suppose $\mathbf{U}_0 \in \mathbb{O}(p, r)$ satisfies $|\mathrm{supp}(\mathbf{U}_0)| \le s$, and $r \le s \le p$. For any positive δ, t, and τ, define*

$$\mathcal{F}(\delta, \tau, t) = \left\{\mathbf{B} \in \mathbb{R}^{p\times r} : |\mathrm{supp}_\delta(\mathbf{B})| \le \tau, \sum_{j=1}^{p} \|\mathbf{B}_{j*}\|_2^2 \mathbb{1}\{j \in \mathrm{supp}_\delta(\mathbf{B}) \cup \mathrm{supp}(\mathbf{U}_0)\} \le t^2\right\}.$$

Let the positive sequences $(\delta_n, \tau_n, t_n, \epsilon_n)_{n=1}^{\infty}$ *satisfy* $(\sqrt{p}\delta_n + 2t_n)\sqrt{p}\delta_n \leq M_1\epsilon_n$ *for some constant* $M_1 > 0$*, and* $\epsilon_n \leq 1$*. Consider testing*

$$H_0 : \mathbf{\Sigma} = \mathbf{\Sigma}_0 = \mathbf{U}_0\mathbf{\Lambda}_0\mathbf{U}_0^{\mathrm{T}} + \sigma_0^2\mathbf{I}_p$$

versus

$$H_1 : \mathbf{\Sigma} \in \left\{\mathbf{\Sigma} = \mathbf{B}\mathbf{B}^{\mathrm{T}} + \sigma^2\mathbf{I}_p : \|\mathbf{\Sigma} - \mathbf{\Sigma}_0\|_2 > M\epsilon_n, \mathbf{B} \in \mathcal{F}(\delta_n, \tau_n, t_n)\right\}.$$

Then for each $M \geq \max\{M_1/2, (128\|\mathbf{\Sigma}_0\|_2^4)^{1/3}\}$*, there exists a test function* $\phi_n : \mathbb{R}^{n\times p} \to [0, 1]$*, such that*

$$\mathbb{E}_0(\phi_n) \leq 3\exp\left\{(2 + C_4)(\tau_n \log p + 2s_n) - \frac{C_4\sqrt{M}}{\sqrt{2}}n\epsilon_n^2\right\},$$
$$\sup_{\mathbf{\Sigma}\in H_1} \mathbb{E}_{\mathbf{\Sigma}}(1 - \phi_n) \leq \exp\left\{C_4(\tau_n + 2s_n) - \frac{C_4 M}{8}n\epsilon_n^2\right\}$$

for some absolute constant $C_4 > 0$*.*

2.5 Numerical examples

2.5.1 Synthetic examples

We evaluate the numerical performance of the proposed Bayesian shrinkage method for estimating sparse spiked covariance matrices via simulation studies. We set the sample size $n = 100$ and the number of features $p = 200$. The support size s of the eigenvector matrix $\mathbf{U}_0$ ranges over $\{8, 12, 20, 40\}$, and the number of spikes r takes values in $\{1, 4\}$. The indices of the non-zero rows of $\mathbf{U}_0$ are uniformly sampled from $\{1, \ldots, p\}$, and we set the diagonal elements of $\mathbf{\Lambda}_0$ to be equally spaced over the interval $[10, 20]$, with $\lambda_{01} = 20$ and $\lambda_{0r} = 10$. The non-zero rows of $\mathbf{U}_0$, themselves forming an orthonormal r-frame in $\mathbb{R}^s$, denoted by $\mathbf{U}_0^{\star}$, are generated as the left singular vector matrix of $\mathbf{L}$, an $s \times r$ matrix consisting of independent $\mathrm{Unif}(1, 2)$ elements.

The posterior inference is implemented using a standard Metropolis-within-Gibbs sampler, and 1000 post-burn-in samples are collected after 1000 iterations of the burn-in phase. We then take the posterior mean $\widehat{\boldsymbol{\Sigma}}$ of $\boldsymbol{\Sigma}$ as the point estimator for $\boldsymbol{\Sigma}$, and the $\widehat{\mathbf{U}}$ given by Theorem 5 as the point estimator for the subspace $\mathrm{Span}\{\mathbf{U}_{*1}, \ldots, \mathbf{U}_{*r}\}$. For comparison, several competitors are considered, including the sparse Bayesian factor model with multiplicative Gamma process shrinkage prior (MGPS, [48]), the principal orthogonal complement thresholding method (POET, [52]), and the sparse principal component analysis method (SPCA, [53]). In each simulation setup (*i.e.*, each (r, s) pair), 50 replicates of synthetic datasets are generated, and for each synthetic dataset, we compute the point estimators $\widehat{\boldsymbol{\Sigma}}$, $\widehat{\mathbf{U}}$ as well as those offered by the three competing approaches, the operator norm loss $\|\widehat{\boldsymbol{\Sigma}} - \boldsymbol{\Sigma}_0\|_2$ for $\boldsymbol{\Sigma}$, the two-to-infinity norm loss and the projection operator norm loss for $\mathrm{Span}\{\mathbf{U}_{*1}, \ldots, \mathbf{U}_{*r}\}$ ($\|\widehat{\mathbf{U}} - \mathbf{U}_0\mathbf{W}_{\mathbf{U}}\|_{2\to\infty}$ and $\|\widehat{\mathbf{U}}\widehat{\mathbf{U}}^{\mathrm{T}} - \mathbf{U}_0\mathbf{U}_0^{\mathrm{T}}\|_2$), and compute the medians of these losses. The results are tabulated in Table 2-I.

The numerical results in Tables 2-I(a) and 2-I(b) indicate that the proposed Bayesian approach yields the smallest operator norm losses for $\boldsymbol{\Sigma}$ and the smallest projection operator norm losses for the subspace estimation, respectively. In terms of the two-to-infinity norm loss for the subspace estimation, Table 2-I(c) shows that the point estimates $\widehat{\mathbf{U}}$ using the proposed approach yield smaller losses compared to the competitors when $s = 8$ and $s = 12$ for both $r = 1$ and $r = 4$, while POET is more accurate for the single-spike cases when $s = 20$ and $s = 40$. The comparison between the two losses for the subspace estimation is also visualized in Figure 2-2, suggesting that the two-to-infinity norm loss is less sensitive to the row support size s than the projection operator norm loss as s increases.

We further evaluate the performance of estimating the principal subspace $\mathrm{Span}\{\mathbf{U}_{*1}, \ldots, \mathbf{U}_{*r}\}$ when $s = 20$, $r = 1$ and $s = 40$, $r = 4$ through a single replicate in Figures 2-3, 2-4, and 2-5, respectively. For visualization of recovering $\mathbf{U}_0$ across different methods, we rotate the estimates according to the Frobenius orthogonal alignment. It can clearly be seen that POET is able to capture the signal but fails

Table 2-I. The operator norm loss $\|\widehat{\boldsymbol{\Sigma}} - \boldsymbol{\Sigma}_0\|_2$ with the posterior mean $\widehat{\boldsymbol{\Sigma}}$, the squared projection operator norm loss $\|\widehat{\mathbf{U}}\widehat{\mathbf{U}}^{\mathrm{T}} - \mathbf{U}_0\mathbf{U}_0^{\mathrm{T}}\|_2^2$, and the squared two-to-infinity norm loss $\|\widehat{\mathbf{U}} - \mathbf{U}_0\mathbf{W}_{\mathbf{U}}\|_{2\to\infty}^2$, where $\widehat{\mathbf{U}}$ is the point estimator of $\mathbf{U}$ given by Theorem 5. The medians across 50 replicates of synthetic datasets are tabulated. MSSL stands for the sparse Bayesian spiked covariance matrix model with the matrix spike-and-slab LASSO prior.

(a) The operator norm loss $\|\widehat{\boldsymbol{\Sigma}} - \boldsymbol{\Sigma}_0\|_2$

s	8		12		20		40	
r	1	4	1	4	1	4	1	4
MSSL	**1.85**	**6.68**	**1.97**	**6.76**	**2.61**	**8.11**	**5.12**	**10.35**
MGPS	9.86	16.54	9.88	17.78	9.88	18.52	9.88	19.05
POET	7.54	11.17	7.47	11.10	7.61	11.60	7.60	10.97
SPCA	8.08	18.03	8.09	18.04	8.11	18.07	8.17	18.10

(b) The squared projection operator norm loss $\|\widehat{\mathbf{U}}\widehat{\mathbf{U}}^{\mathrm{T}} - \mathbf{U}_0\mathbf{U}_0^{\mathrm{T}}\|_2^2$

s	8		12		20		40	
r	1	4	1	4	1	4	1	4
MSSL	**0.0099**	**0.033**	**0.018**	**0.036**	**0.026**	**0.046**	**0.10**	**0.061**
MGPS	0.18	0.27	0.19	0.47	0.20	0.35	0.20	0.27
POET	0.18	0.21	0.18	0.20	0.19	0.20	0.18	0.20
SPCA	0.05	0.092	0.068	0.11	0.10	0.15	0.18	0.22

(c) The squared two-to-infinity norm loss $\|\widehat{\mathbf{U}} - \mathbf{U}_0\mathbf{W}_{\mathbf{U}}\|_{2\to\infty}^2$

s	8		12		20		40	
r	1	4	1	4	1	4	1	4
MSSL	**0.0038**	**0.011**	**0.0058**	**0.012**	0.014	**0.012**	0.016	**0.011**
MGPS	0.0093	0.085	0.0096	0.14	0.0092	0.14	0.01	0.077
POET	0.0082	0.013	0.0082	0.013	**0.0086**	0.012	**0.0088**	0.013
SPCA	0.024	0.027	0.022	0.040	0.022	0.039	0.025	0.038

to recover the joint sparsity of the principal subspace, whereas SPCA can recover the subspace sparsity but is not accurate in estimating the signal. MGPS performs similarly to POET, but its estimated credible intervals are wider than those using the proposed approach.

Overall, the proposed sparse Bayesian spiked covariance matrix model can estimate the signals accurately, recover the row support of $\mathbf{U}_0$, and provides better uncertainty quantification with narrower credible intervals for simulation setting.

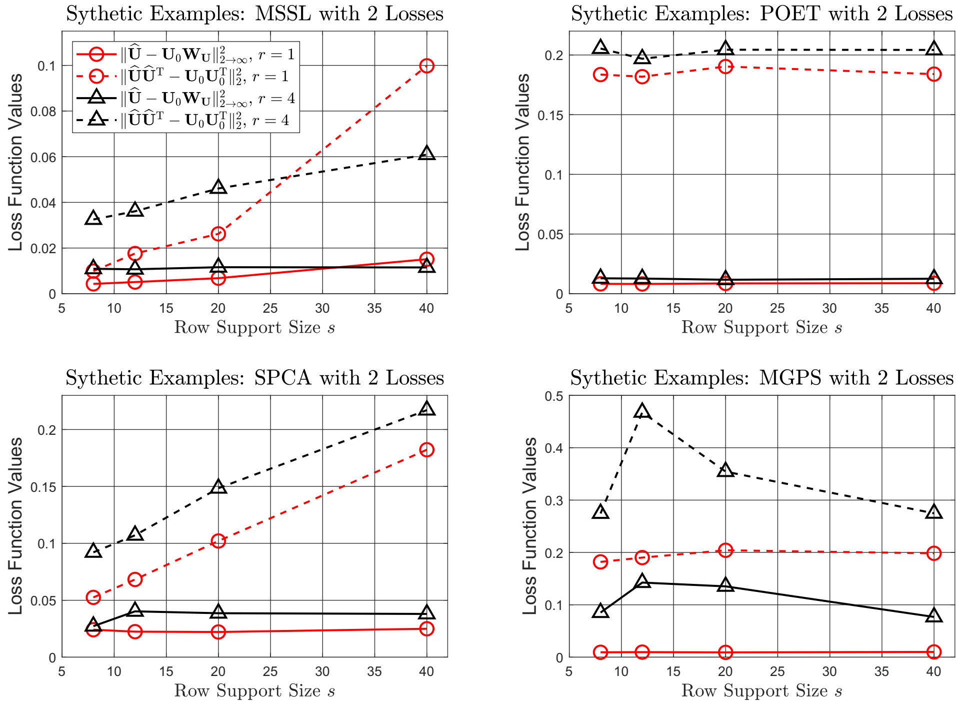


Figure 2-2. Comparison of the two-to-infinity norm loss ($\|\widehat{\mathbf{U}}-\mathbf{U}_0\mathbf{W}_{\mathbf{U}}\|_{2\to\infty}$) and the projection operator norm loss ($\|\widehat{\mathbf{U}}\widehat{\mathbf{U}}^{\mathrm{T}}-\mathbf{U}_0\mathbf{U}_0^{\mathrm{T}}\|_2$) for synthetic examples. MSSL stands for the sparse Bayesian spiked covariance matrix model with the matrix spike-and-slab LASSO prior.

2.5.2 A face data example

The joint sparsity of columns of the eigenvector matrix $\mathbf{U}$ is highly desired in feature extraction for high-dimensional data. In this subsection, we illustrate how the proposed Bayesian approach is able to extract key features through a real data example in computer vision.

We consider a subset of the Extended Yale Face Database B [5, 6]. It consists of face images for 38 subjects, and for each subject, 64 aligned images of size 192×168 are taken under different illumination conditions. Here we focus on the 22nd subject and reduce the size of each image to 96×84 (8064 pixels in total), following [54]. In doing so we obtain a data matrix $\mathbf{Y} = [\mathbf{y}_1, \ldots, \mathbf{y}_n]^{\mathrm{T}}$ of size 64×8064.

In computer vision, principal component analysis has been broadly applied to

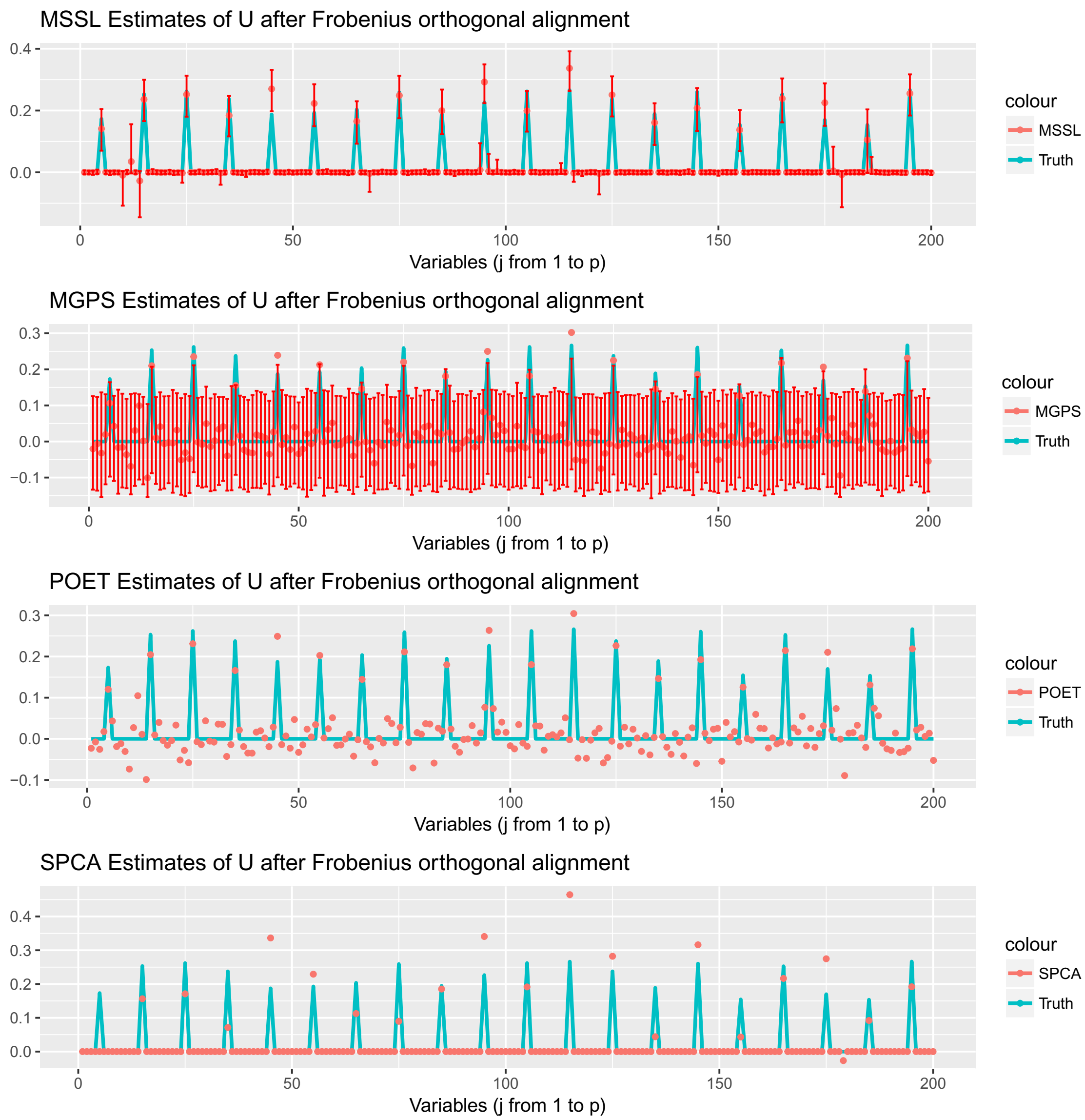


Figure 2-3. Simulation performance from a single replicate with $s = 20$ and $r = 1$. The estimates are rotated to the simulation truth $\mathbf{U}_0$ according to the Frobenius orthogonal alignment. The red bars in the top panels are estimated 95% credible intervals using the proposed approach. MSSL stands for the sparse Bayesian spiked covariance matrix model with the matrix spike-and-slab LASSO prior.

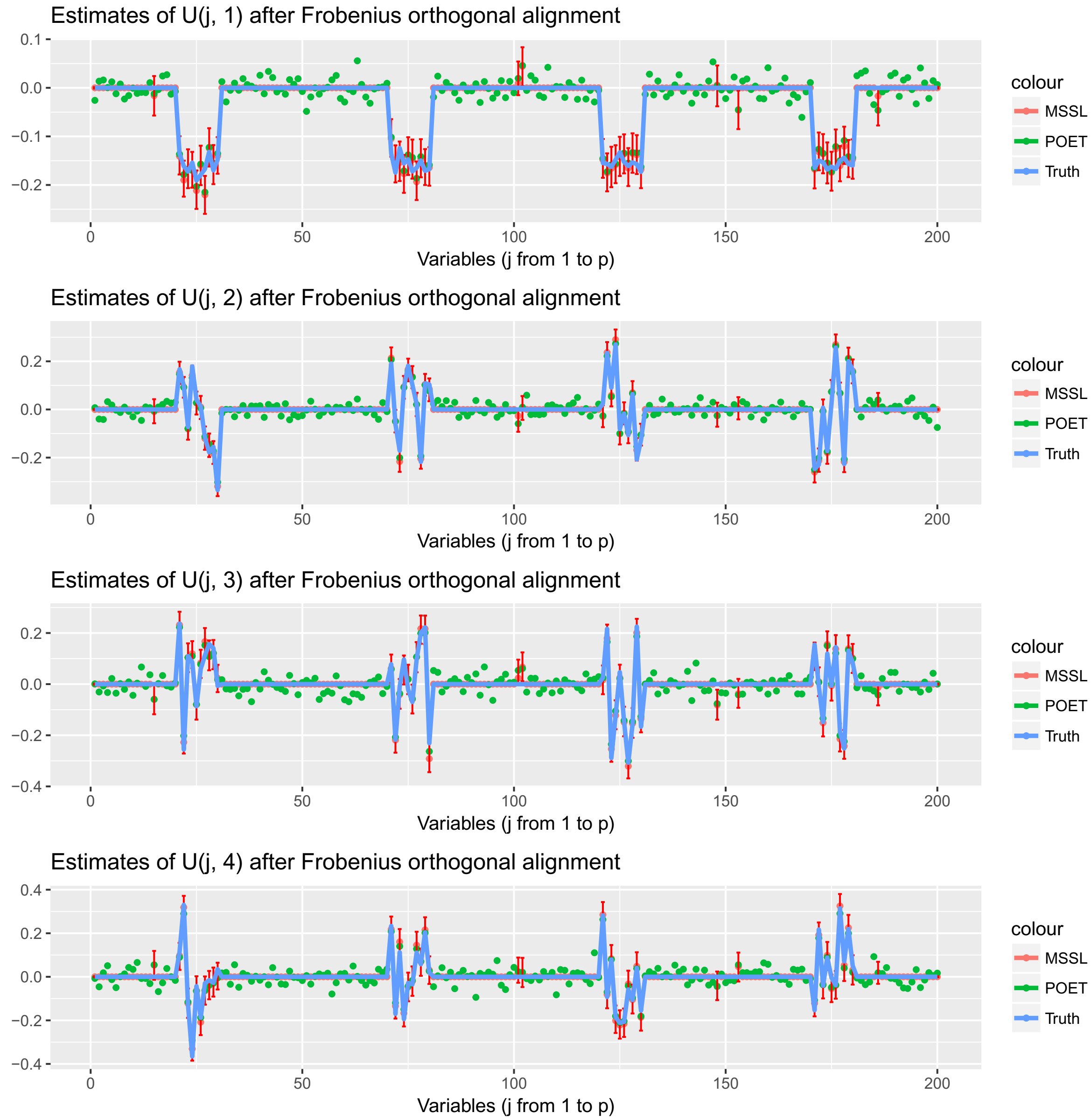


Figure 2-4. Simulation performance from a single replicate with $s = 40$ and $r = 4$. The estimates are rotated to the simulation truth $\mathbf{U}_0$ according to the Frobenius orthogonal alignment. The red bars in the four panels are estimated 95% credible intervals using the proposed approach.

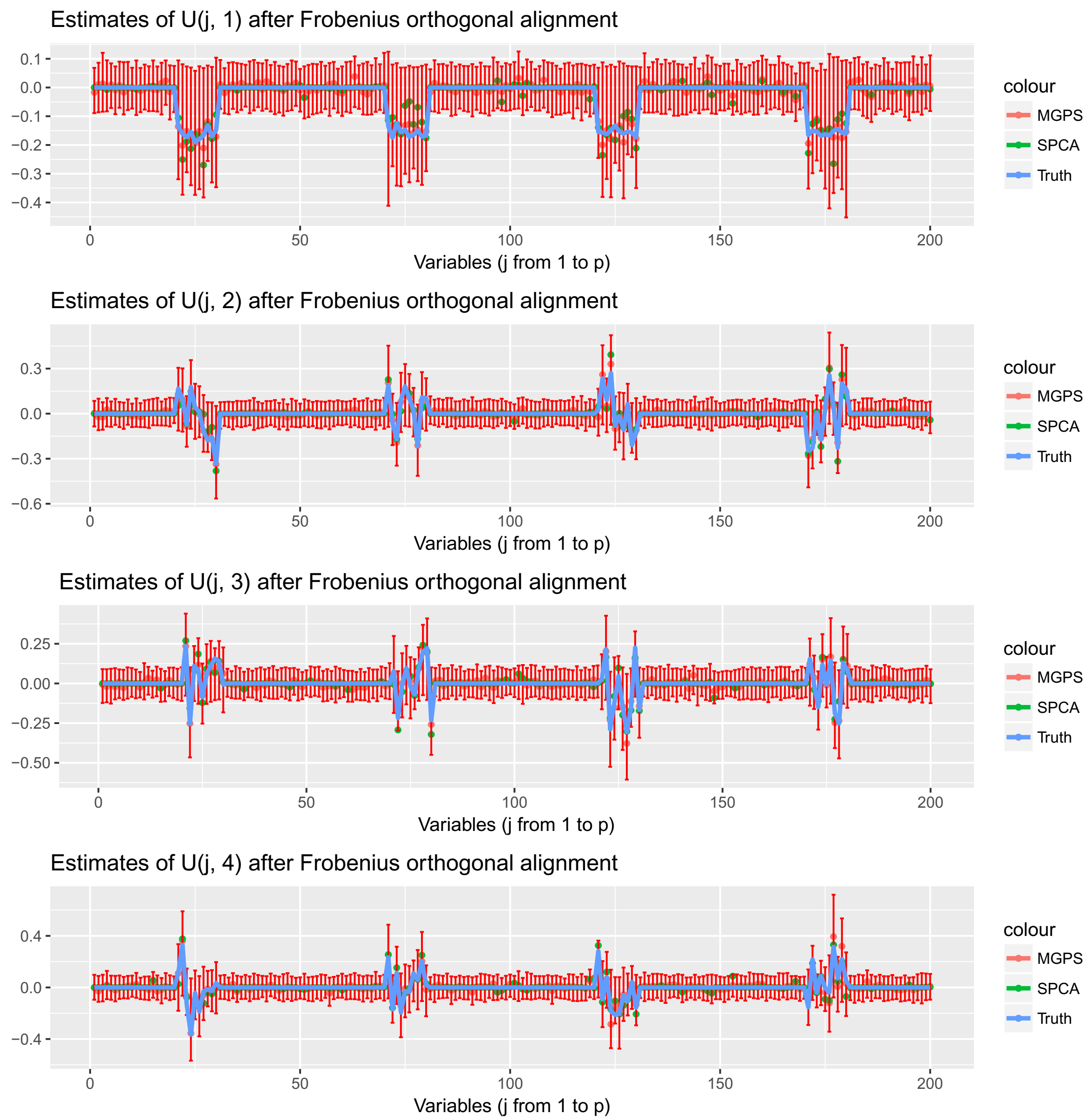


Figure 2-5. Simulation performance from a single replicate with $s = 40$ and $r = 4$. The estimates are rotated to the simulation truth $\mathbf{U}_0$ according to the Frobenius orthogonal alignment. The red bars in the four panels are estimated 95% credible intervals for MGPS.

obtain low-dimensional features, known as eigenfaces, from high-dimensional face image data. Under the proposed hierarchical Bayesian model, we perform posterior inference by implementing a Metropolis-within-Gibbs sampler. The number of spikes r is estimated using the diagonal thresholding method proposed in [28]. For comparison, we also implement MGPS [48]. Instead of obtaining eigenfaces, we focus on directly extracting the key pixels via thresholding the obtained estimated eigenvector matrix $\widehat{\mathbf{U}}$ using the obtained posterior samples. Specifically, for the proposed approach, the estimate $\widehat{\mathbf{U}}$ can be computed according to Theorem 5, and for MGPS, $\widehat{\mathbf{U}}$ can be obtained by computing the left singular vectors of the loading matrix. The key pixels are then obtained by finding $\{j \in [8064] : \|\widehat{\mathbf{U}}_{j*}\|_1/r > \tau\}$ for some small tolerance $\tau > 0$.

We present sample images of the 22nd subject in the first row of Figure 2-6, and the key pixels of the sample image #1 extracted under the two models with different threshold values of τ are provided in the second and the third rows of Figure 2-6. Under both models, pixels with higher values (corresponding to eyes, cheeks, forehead, and nose tips of the subject) are recovered. This observation is also in accordance with the conclusion from [54]. Nevertheless, as the threshold value τ increases, the number of key pixels captured using MGPS decreases significantly, whereas the proposed approach is more robust to the threshold value τ and maintains the key pixels that are sensitive to illumination. This phenomenon is expected since MGPS is not designed to model joint sparsity and feature extraction, but rather column-specific sparsity for each individual factor loading, unlike the matrix spike-and-slab LASSO prior.

2.6 Discussion

We have shown that the two-to-infinity norm loss for principal subspace estimation is superior to the routinely used projection operator norm loss in that the former is able to capture element-wise perturbations of the eigenvector matrix $\mathbf{U}$ compared to

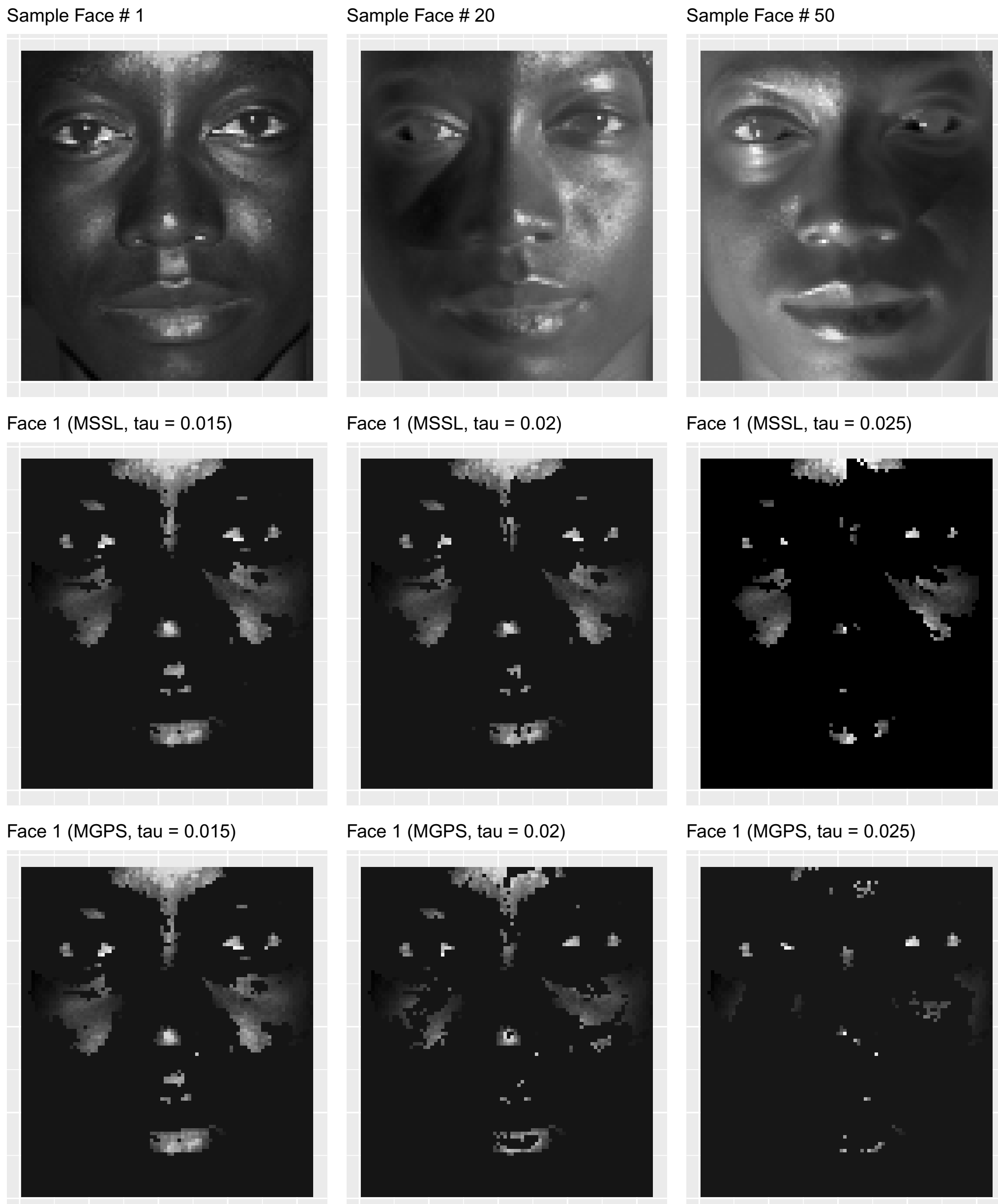


Figure 2-6. The face data example: The first row corresponds to sample images of the 22nd subject (image number 1, 20, and 50, respectively). The second and the third rows are the key pixels of the $\#1$ image using the proposed Bayesian approach with the matrix spike-and-slab LASSO prior (MSSL) and MGPS with different threshold values of τ.

the latter. We have derived the contraction rate of the full posterior distribution for the principal subspace with respect to the two-to-infinity norm loss, which is tighter than that with respect to the usual projection operator norm loss, provided that $\mathbf{U}$ exhibits certain low-rank and bounded coherence features. In future work, we intend to study whether a point estimator can be found from the posterior distribution with a risk bound that coincides with the posterior contraction rate with respect to the two-to-infinity norm loss. In addition, it is also worth exploring the minimax-optimal rates of convergence with respect to the two-to-infinity norm loss.

Throughout the chapter, the number of spikes r is either assumed to be known or unknown but can be consistently estimated using a frequentist procedure. Alternatively, it is feasible to adaptively estimate r in the literature of Bayesian latent factor models (see, for example, [48, 40, 41]). Hence exploring rank-adaptive Bayesian procedure and obtain attractive theoretical properties or computation tractability could also be interesting.

Markov chain Monte Carlo (MCMC) can be computationally intensive for high-dimensional settings in general. In this chapter, we explored MCMC for Bayesian estimation of the sparse spiked covariance matrix models. It would be attractive to design efficient computational methods, such as the expectation-maximization algorithm for the maximum *a posteriori* estimation instead of computing the full posterior distribution [55], or penalized least-squares estimation [54], and explore the underlying theoretical guarantees in future work.

Chapter 3

Random dot product graphs: Optimal global estimation

3.1 Motivation and overview

The techniques for statistical analysis of the random dot product graph model so far have been focusing on spectral methods based on the observed adjacency matrix or its graph Laplacian matrix. For example, the authors of [32] proposed to estimate the latent positions using the adjacency spectral embedding directly and proved its consistency. For the normalized graph Laplacian matrix of the adjacency matrix, the authors of [56] found the asymptotic distribution of spectral embedding using the normalized graph Laplacian. They made a thorough comparison between the adjacency spectral embedding and the Laplacian spectral embedding under various contexts. As mentioned earlier in Section 1.4, the well-developed theory for spectral methods for the random dot product graph model lays a theoretical foundation for a variety of subsequent inference tasks. Despite the marvelous success of spectral methods for the random dot product graph model, it remains open whether these spectral estimators are minimax-optimal for estimating the latent positions with respect to suitable loss functions. Taking one step back, we are interested in a more fundamental question: What is the minimax risk for estimating the latent positions, and how can one achieve it by constructing a useful estimator? In this chapter, we provide a detailed answer to this question. Unlike the aforementioned spectral-based approaches, we take advantage of the Bernoulli likelihood information of the observed graph adjacency matrix and design a fully likelihood-based Bayesian approach, referred to as *posterior spectral embedding*. Not only do we establish a minimax lower bound for estimating the latent positions, but we also show that this lower bound is achievable through the proposed Bayes procedure. Specifically, we show that the posterior spectral embedding both yields the rate-optimal contraction and produces a minimax-optimal point estimator for estimating the latent positions.

Recall the loss function

$$L(\widehat{\mathbf{X}}, \mathbf{X}_0) = \frac{1}{n} \inf_{\mathbf{W} \in \mathbb{O}(d)} \|\widehat{\mathbf{X}} - \mathbf{X}_0 \mathbf{W}\|_{\mathrm{F}}^2$$

for any embedding estimator $\widehat{\mathbf{X}}$ and the true latent position matrix $\mathbf{X}_0$. We first present the following minimax lower bound, which can be regarded as a metric for measuring the performance of various estimators, before elaborating on the proposed Bayesian approach for embeddings. In particular, an estimator $\widehat{\mathbf{X}}$ for the latent position matrix $\mathbf{X}_0$ is said to be *globally optimal*, if $\mathbb{E}_0\{L(\widehat{\mathbf{X}}, \mathbf{X}_0)\}$ achieves the minimax lower bound asymptotically up to a multiplicative constant.

Theorem 6 *Let* $\mathbf{A} \sim \mathrm{RDPG}(\mathbf{X})$ *for some* $\mathbf{X} = [\mathbf{x}_1, \ldots, \mathbf{x}_n]^{\mathrm{T}}$, $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathcal{X}$. *Assume that* d *is fixed and does not change with* n. *Let* $\widehat{\mathbf{X}}$ *be an estimator of the latent position matrix* $\mathbf{X}$ *satisfying* $\|\widehat{\mathbf{X}}\|_{\mathrm{F}} \lesssim n^{1/2}$ *with probability one. Then*

$$\inf_{\widehat{\mathbf{X}}} \sup_{\mathbf{X} \in \mathcal{X}^n} \mathbb{E}_{\mathbf{X}} \left(\frac{1}{n} \inf_{\mathbf{W} \in \mathbb{O}(d)} \|\widehat{\mathbf{X}} - \mathbf{X}\mathbf{W}\|_{\mathrm{F}}^2 \right) \gtrsim \frac{1}{n}. \tag{3.1}$$

The rest of this chapter is organized as follows. Section 3.2 elaborates on the proposed likelihood-based Bayesian method, referred to as the *posterior spectral embedding* for the random dot product graph model and its theoretical properties. An easy-to-implement Metropolis-Hastings sampler for posterior computation is provided in Section 3.3, and an application to Bayesian clustering in stochastic block models is discussed in Section 3.4. Section 3.5 presents the analysis of a spectral-based Gaussian spectral embedding approach that can be treated as a Bayesian analogy of the adjacency spectral embedding. We extend the current framework to sparse and directed networks in Section 3.6 and illustrate the proposed approach through extensive simulation studies in Section 3.7. Further discussion is provided in Section 3.8.

3.2 The posterior spectral embedding

As discussed in Section 1.4, although it is intuitive and computationally convenient to directly estimate the latent position matrix $\mathbf{X}$ by the popular spectral-based approaches, *e.g.*, the adjacency spectral embedding, the Bernoulli likelihood information of the adjacency matrix is neglected. On the other hand, likelihood-based methods for the random dot product graph model remain under-explored. In particular, neither the existence nor the uniqueness of the maximum likelihood estimator for $\mathbf{X}$ has been addressed. In this section, we develop a Bayesian approach for estimating the latent positions by taking advantage of the Bernoulli likelihood information.

Recall that the space of the latent positions is $\mathcal{X} = \{\mathbf{x} \in \mathbb{R}^d : \|\mathbf{x}\|_2 < 1, \mathbf{x} > \mathbf{0}\}$. Let $\mathbf{X}_0 = [\mathbf{x}_{01}, \ldots, \mathbf{x}_{0n}]^{\mathrm{T}}$ be the true latent position matrix, and $\mathbf{X} = [\mathbf{x}_1, \ldots, \mathbf{x}_n]^{\mathrm{T}}$ be the latent position matrix to be assigned a prior distribution Π. Whenever we consider the distribution Π, $\mathbf{X}$ is treated as a random matrix taking values in the space $\mathcal{X}^n = \{\mathbf{X} = [\mathbf{x}_1, \ldots, \mathbf{x}_n]^{\mathrm{T}} : \mathbf{x}_i \in \mathcal{X}, i = 1, \ldots, n\}$. The prior distribution Π on $\mathbf{X}$ is constructed by assuming that $\mathbf{x}_1, \ldots, \mathbf{x}_n$ follow a distribution with a density function $\pi_{\mathbf{x}}$ supported on $\mathcal{X}$ independently, and we denote it by $\mathbf{X} \sim \Pi$. Namely,

$$\Pi(\mathrm{d}\mathbf{X}) = \prod_{i=1}^{n} \pi_{\mathbf{x}}(\mathbf{x}_i)\mathrm{d}\mathbf{x}_i.$$

In this chapter, we only require $\pi_{\mathbf{x}}$ to be bounded away from 0 and ∞ over $\mathcal{X}$, *e.g.*, the uniform distribution on $\mathcal{X}$. It follows directly from the Bayes formula that the posterior distribution of $\mathbf{X}$ is

$$\Pi(\mathbf{X} \in \mathcal{A} \mid \mathbf{A}) = \frac{N_n(\mathcal{A})}{D_n}, \quad N_n(\mathcal{A}) = \int_{\mathcal{A}} \prod_{i \leq j} \frac{p(A_{ij} \mid \mathbf{X})}{p(A_{ij} \mid \mathbf{X}_0)} \Pi(\mathrm{d}\mathbf{X}), \quad D_n = N_n(\mathcal{X}),$$

and $p(A_{ij} \mid \mathbf{X}) = (\mathbf{x}_i^{\mathrm{T}}\mathbf{x}_j)^{A_{ij}}(1 - \mathbf{x}_i^{\mathrm{T}}\mathbf{x}_j)^{1-A_{ij}}$, for any measurable set $\mathcal{A} \subset \mathcal{X}^n$. Clearly, the posterior distribution of $\mathbf{X}$ incorporates the Bernoulli likelihood information through the Bayes formula, and we refer to $\Pi(\mathbf{X} \in \cdot \mid \mathbf{A})$ as the *posterior spectral embedding*.

The following theorem, which is the key result of this work, shows that under mild regularity conditions, the posterior contraction rate of the posterior spectral embedding is minimax-optimal. The proof is deferred to Chapter 5.

Theorem 7 *Let* $\mathbf{A} \sim \mathrm{RDPG}(\mathbf{X}_0)$ *for some* $\mathbf{X}_0 = [\mathbf{x}_{01}, \ldots, \mathbf{x}_{0n}]^{\mathrm{T}} \in \mathbb{R}^{n \times d}$*, and the prior* Π *be described as above. Assume that* $(1/n)(\mathbf{X}_0^{\mathrm{T}}\mathbf{X}_0) \to \mathbf{\Delta}$ *as* $n \to \infty$ *for some positive definite* $\mathbf{\Delta} \in \mathbb{R}^{d \times d}$*. If* d *is fixed, and* $\delta \leq \min_{i,j} \mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j} \leq \max_{i,j} \mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j} \leq 1 - \delta$ *for some constant* $\delta \in (0, 1/2)$ *independent of* n*, then there exist some large constants* $M_1, M_2 > 0$ *depending on* $\mathbf{\Delta}$ *and the prior* $\pi_{\mathbf{x}}$*, such that*

$$\mathbb{E}_0\left\{\Pi\left(\frac{1}{n}\|\mathbf{X}\mathbf{X}^{\mathrm{T}} - \mathbf{X}_0\mathbf{X}_0^{\mathrm{T}}\|_{\mathrm{F}} > \frac{M_1}{\sqrt{n}} \;\middle|\; \mathbf{A}\right)\right\} \leq 8\exp\left(-\frac{1}{2}nd\right),$$

$$\mathbb{E}_0\left\{\Pi\left(\frac{1}{n}\inf_{\mathbf{W} \in \mathbb{O}(d)} \|\mathbf{X} - \mathbf{X}_0\mathbf{W}\|_{\mathrm{F}}^2 > \frac{M_2}{n} \;\middle|\; \mathbf{A}\right)\right\} \leq 8\exp\left(-\frac{1}{2}nd\right)$$

for sufficiently large n*.*

Remark 5 *The assumption* $(1/n)(\mathbf{X}_0^{\mathrm{T}}\mathbf{X}_0) \to \mathbf{\Delta}$ *as* $n \to \infty$ *in Theorem 7 can be equivalently written as* $(1/n)\sum_{i=1}^n \mathbf{x}_{0i}\mathbf{x}_{0i}^{\mathrm{T}} \to \mathbf{\Delta}$ *as* $n \to \infty$ *for some positive definite* $\mathbf{\Delta}$*. An intuitive interpretation of this condition is that the true latent positions* $\mathbf{x}_{01}, \ldots, \mathbf{x}_{0n}$ *can be regarded as "random" samples drawn from some non-degenerate distribution with a positive definite second-moment matrix* $\mathbf{\Delta}$*. By the law of large numbers, the "sample" version of the second-moment matrix "converges" to the "population" version of the second-moment matrix. An illustrative example is the positive semidefinite stochastic block model: Suppose the distinct latent positions of* $\mathbf{x}_{01}, \ldots, \mathbf{x}_{0n}$ *are* $\mathbf{x}_{01}^*, \ldots, \mathbf{x}_{0K}^*$*,* *and let $n_k = \sum_{i=1}^n \mathbb{1}(\mathbf{x}_{0i} = \mathbf{x}_{0k}^*)$ be the number of vertices corresponding to the latent position $\mathbf{x}_{0k}^*$. Assume that K is fixed, $n_k/n \to \alpha_k > 0$ as $n \to \infty$, and α_k's, $\mathbf{x}_{0k}^*$'s are fixed for $k = 1, \ldots, K$. Then*

$$\frac{1}{n}\mathbf{X}_0^{\mathrm{T}}\mathbf{X}_0 = \sum_{k=1}^K \sum_{i=1}^n \mathbb{1}(\mathbf{x}_{0i} = \mathbf{x}_{0k}^*)\mathbf{x}_{0i}\mathbf{x}_{0i}^{\mathrm{T}} = \sum_{k=1}^K \frac{n_k}{n}(\mathbf{x}_{0k}^*)(\mathbf{x}_{0k}^*)^{\mathrm{T}} \to \sum_{k=1}^K \alpha_k(\mathbf{x}_{0k}^*)(\mathbf{x}_{0k}^*)^{\mathrm{T}}$$

as $n \to \infty$. Therefore, with the above assumption, the positive semidefinite stochastic block model satisfies this condition provided that $\sum_{k=1}^K \alpha_k(\mathbf{x}_{0k}^*)(\mathbf{x}_{0k}^*)^{\mathrm{T}}$ is positive definite.

Theorem 7 claims that under appropriate regularity conditions, the posterior spectral embedding yields the rate-optimal posterior contraction for the latent positions in the Bayesian sense. The following theorem shows that one can use the posterior spectral embedding to construct a point estimator $\widehat{\mathbf{X}}$ further that exactly achieves the minimax lower bound (3.1) in the classical frequentist sense.

Theorem 8 *Let the conditions in Theorem 7 hold, and let constant $M_1 > 0$ be given by Theorem 7. Consider the posterior mean of the edge probability matrix*

$$\widetilde{\mathbf{P}} = \int_{\mathbf{X}\in\mathcal{X}^n} \mathbf{X}\mathbf{X}^{\mathrm{T}}\Pi(\mathrm{d}\mathbf{X} \mid \mathbf{A}).$$

Suppose $\widetilde{\mathbf{P}}$ yields spectral decomposition $\widetilde{\mathbf{P}} = \sum_{j=1}^n \widehat{\lambda}_j\widehat{\mathbf{u}}_j$, where $\widehat{\lambda}_1, \ldots, \widehat{\lambda}_n$ are eigenvalues of $\widetilde{\mathbf{P}}$ arranged in non-increasing order, and $\widehat{\mathbf{u}}_1, \ldots, \widehat{\mathbf{u}}_n$ are the associated eigenvectors. Let $\widehat{\mathbf{U}} = (\widehat{\mathbf{u}}_1, \ldots, \widehat{\mathbf{u}}_d)$, $\widehat{\mathbf{S}} = \mathrm{diag}(\widehat{\lambda}_1, \ldots, \widehat{\lambda}_d)$, $\widehat{\mathbf{X}} = \widehat{\mathbf{U}}\widehat{\mathbf{S}}^{1/2}$, and $\mathbf{U}_0$ be the left-singular vector matrix of $\mathbf{X}_0$. Then for sufficiently large n,

$$\mathbb{E}_0\left(\frac{1}{n}\inf_{\mathbf{W}\in\mathbb{O}(d)}\|\widehat{\mathbf{X}} - \mathbf{X}_0\mathbf{W}\|_{\mathrm{F}}^2\right) \lesssim \frac{1}{n}. \tag{3.2}$$

Furthermore, for sufficiently large n,

$$\mathbb{P}_0\left\{\inf_{\mathbf{W}\in\mathbb{O}(d)}\|\widehat{\mathbf{U}} - \mathbf{U}_0\mathbf{W}\|_{\mathrm{F}}^2 > \frac{128M_1^2 d}{\lambda_d^2(\boldsymbol{\Delta})n}\right\} \leq 2\exp\left(-\frac{1}{4}M_1 d\sqrt{n}\right). \tag{3.3}$$

We briefly compare the results of Theorem 8 with those in [32]. The convergence rate (3.2) shows that $\widehat{\mathbf{X}}$ achieves the minimax lower bound (3.1). The convergence rate of the unscaled eigenvectors $\widehat{\mathbf{U}}$ given by (3.3) also improves its counterpart in [32], which is explained as follows. Denote $\mathbf{U}$ the left-singular vector matrix of $\mathbf{X}$, and $\widehat{\mathbf{U}}_{\mathrm{ASE}}$ that of $\widehat{\mathbf{X}}_{\mathrm{ASE}}$. Then under the conditions of [32] (which is stronger than the conditions of Theorem 8), there exists an orthogonal matrix $\mathbf{W}$ such that

$$\mathbb{P}_0\left\{\|(\widehat{\mathbf{U}}_{\mathrm{ASE}})_{*k} - (\mathbf{W}\mathbf{U}_0)_{*k}\|_2^2 > \frac{3\log n}{\delta^2 n}\right\} \leq \frac{2(d^2+1)}{n^2} \tag{3.4}$$

for $k = 1, \ldots, d$. In contrast, the eigenvector estimate $\widehat{\mathbf{U}}$ derived using the posterior spectral embedding improves the convergence rate (3.4): Not only do we improve the rate from $(\log n)/n$ to $1/n$, but we also sharpen the large deviation probability from $O(1/n^2)$ to $O(\mathrm{e}^{-cn^{1/2}})$ for some constant $c > 0$. The distinct eigenvalues condition for $\boldsymbol{\Delta}$ required in [32] is also dropped.

3.3 The Metropolis-Hastings sampler

Besides theoretical guarantee in terms of global optimality, another fascinating feature of the posterior spectral embedding is that the computation can be carried out using an easy-to-implement Metropolis-Hastings sampler. Specifically, we detailed the Metropolis-Hastings sampler for the posterior spectral embedding in this section. The sampler is initialized with randomly sampled latent positions $\mathbf{x}_1^{(1)}, \ldots, \mathbf{x}_n^{(1)}$ that are uniformly drawn from $\mathcal{X}$. The following theorem shows that one can obtain $\mathrm{Unif}(\mathcal{X})$ random vectors by a smart transformation of a collection of Gamma random variables.

Theorem 9 *Let $v_1, \ldots, v_d \sim \mathrm{Gamma}(1/2, 1)$ independently, and $v_{d+1} \sim \mathrm{Gamma}(1, 1)$, independent of $v_1, \ldots, v_d$. Then the random vector*

$$\left[\sqrt{\frac{v_1}{\sum_{k=1}^{d+1} v_k}}, \ldots, \sqrt{\frac{v_d}{\sum_{k=1}^{d+1} v_k}}\right]^{\mathrm{T}}$$

follows the uniform distribution on $\mathcal{X}$.

Proof. Denote $w_k = v_k / \sum_{\ell=1}^{d+1} v_\ell$, $k = 1, \ldots, d$. Clearly, the random vector $\mathbf{w} :=$

$[w_1, \ldots, w_d, 1 - \sum_{k=1}^d w_k]^{\mathrm{T}}$ follows Dirichlet$(1/2, \ldots, 1/2, 1)$ in $\mathbb{R}^{d+1}$, namely,

$$p_{\mathbf{w}}(w_1, \ldots, w_d) \propto \prod_{k=1}^d w_k^{-1/2}.$$

Now consider the change of variables $\mathbf{x} := [x_1, \ldots, x_d]^{\mathrm{T}} = [\sqrt{w_1}, \ldots, \sqrt{w_d}]^{\mathrm{T}}$. Clearly, the corresponding Jacobian is $\partial \mathbf{w}/\partial \mathbf{x} = \mathrm{diag}(2x_1, \ldots, 2x_d)$. It follows that the density function of $\mathbf{x}$ is

$$p_{\mathbf{x}}(x_1, \ldots, x_d) = p_{\mathbf{w}}(x_1^2, \ldots x_d^2) \det\left(\frac{\partial \mathbf{w}}{\partial \mathbf{x}}\right) \propto \left\{\prod_{k=1}^d (x_k^2)^{-1/2}\right\}\left\{\prod_{k=1}^d (2x_k)\right\} \propto 1.$$

This shows that the random vector

$$\mathbf{x} = \left[\sqrt{\frac{v_1}{\sum_{k=1}^{d+1} v_k}}, \ldots, \sqrt{\frac{v_d}{\sum_{k=1}^{d+1} v_k}}\right]^{\mathrm{T}}$$

has a density function that is proportional to 1, and thus follows the uniform distribution on $\mathcal{X}$. $\square$

We now provide the detailed Metropolis-Hastings sampler for the posterior spectral embedding in Algorithm 1 below.

Algorithm 1 The Metropolis-Hastings sampler for the posterior spectral embedding

```
Input:
    Adjacency matrix A = [A_ij]_{n×n};
    embedding dimension d.
User define:
    Number of burn-in iterations B;
    Number of post-burn-in samples n_mc;
    thinning size b.
Initialize:
    For i = 1, ..., n
        Draw v_{i1}^{(1)}, ..., v_{id}^{(1)} independently from Gamma(1/2, 1)
        Draw v_{i(d+1)}^{(1)} from Gamma(1, 1), independent of v_{i1}, ..., v_{id}
        For k = 1, ..., d
            Set x_{ik}^{(1)} = sqrt( v_{ik}^{(1)} / sum_{r=1}^{k} v_{ir}^{(1)} )
        End For
    End For
```

Start sampling:

For $t = 2$ to $B + n_{\mathrm{mc}} \times b$

For $i = 1$ to n

Set

$$\ell_{\mathrm{old}}^{(i)} = \sum_{j=1}^{n} A_{ij} \log\left\{(\mathbf{x}_i^{(t-1)})^{\mathrm{T}}(\mathbf{x}_j^{(t-1)})\right\} + \sum_{j=1}^{n}(1 - A_{ij}) \log\left\{1 - (\mathbf{x}_i^{(t-1)})^{\mathrm{T}}(\mathbf{x}_j^{(t-1)})\right\}$$

For $k = 1$ to $d + 1$

Draw $v_{ik}^* \sim q_k(v)$

For $r = 1$ to d

Set $x_{ir}^* = v_{ir}^{(t-1)} / \sqrt{\sum_{s=1, s\neq k}^{d+1} (v_{is}^{(t-1)})^2 + (v_{ik}^*)^2}$ if $r \neq k$

Set $x_{ir}^* = v_{ik}^* / \sqrt{\sum_{s=1, s\neq k}^{d+1} (v_{is}^{(t-1)})^2 + (v_{ik}^*)^2}$ if $r = k$

End For

Set $\mathbf{x}_i^* = [x_{i1}^*, \ldots, x_{id}^*]^{\mathrm{T}}$

Set

$$\ell_{\mathrm{new}}^{(i)} = \sum_{j=1}^{n} A_{ij} \log\left\{(\mathbf{x}_i^*)^{\mathrm{T}}(\mathbf{x}_j^{(t-1)})\right\} + \sum_{j=1}^{n}(1 - A_{ij}) \log\left\{1 - (\mathbf{x}_i^*)^{\mathrm{T}}(\mathbf{x}_j^{(t-1)})\right\}$$

Draw $U \sim \mathrm{Unif}(0, 1)$

Compute

$$\log \rho = \ell_{\mathrm{new}}^{(i)} - \ell_{\mathrm{old}}^{(i)} - \sum_{r=1, r\neq k}^{d+1} \log q_r(v_{ir}^{(t-1)}) - q_k(v_{ik}^*) + \sum_{r=1}^{d+1} \log q_r(v_{ir}^{(t-1)})$$

If $\log U < \log \rho$

Set $v_{ik}^{(t)} = v_{ik}^*$

Else

Set $v_{ik}^{(t)} = v_{ik}^{(t-1)}$

End if

End For

For $k = 1$ to d

Set $x_{ik}^{(t)} = v_{ik}^{(t)} / \sqrt{\sum_{r=1}^{d+1} (v_{ik}^{(t)})^2}$

End For

End For

End For

Output: $\{\mathbf{X}^{(B+1+bN)}\}_{N=1}^{\lceil (n_{\mathrm{mc}}-1)/b \rceil}$, where $\mathbf{X}^{(t)} = [\mathbf{x}_1^{(t)}, \ldots, \mathbf{x}_n^{(t)}]^{\mathrm{T}}$

3.4 Clustering in stochastic block models

In a broad range of real-world network data analysis, obtaining embedding estimates for the latent positions are typically the first foundation step that can be interpreted as dimensionality reduction of network data. These embedding estimates are often applied for some subsequent inference tasks. This section presents an application of the posterior spectral embedding to clustering in (positive semidefinite) stochastic block models.

We first review the K-means clustering procedure in general [57] before presenting the proposed clustering method. Suppose that n data points $\widehat{\mathbf{x}}_1, \ldots, \widehat{\mathbf{x}}_n$ in $\mathbb{R}^d$ are to be assigned into K clusters, and denote $\widehat{\mathbf{X}} = [\widehat{\mathbf{x}}_1, \ldots, \widehat{\mathbf{x}}_n]^{\mathrm{T}} \in \mathbb{R}^{n\times d}$ the corresponding data matrix. The K-means clustering centroids of $\widehat{\mathbf{x}}_1, \ldots, \widehat{\mathbf{x}}_n$, represented by an $n \times d$ matrix $C(\widehat{\mathbf{X}})$ with K distinct rows, are given by

$$\mathbf{C}(\widehat{\mathbf{X}}) = \underset{\mathbf{C}\in\mathcal{C}_K}{\arg\min}\, \|\mathbf{C} - \widehat{\mathbf{X}}\|_{\mathrm{F}}, \quad \text{where} \quad \mathcal{C}_K = \{\mathbf{C} \in \mathbb{R}^{n\times d} : \mathbf{C} \text{ has } K \text{ distinct rows}\}.$$

The corresponding cluster assignment function is defined to be any function $\tau(\cdot; \widehat{\mathbf{X}}) : [n] \to [K]$ such that $\tau(i; \widehat{\mathbf{X}}) = \tau(j; \widehat{\mathbf{X}})$ if and only if $\{\mathbf{C}(\widehat{\mathbf{X}})\}_{i*} = \{\mathbf{C}(\widehat{\mathbf{X}})\}_{j*}$. Given two cluster assignment functions $\tau_1, \tau_2 : [n] \to [K]$, the Hamming distance between τ_1 and τ_2 is defined by $d_H(\tau_1, \tau_2) = \sum_{i=1}^{n} \mathbb{1}\{\tau_1(i) \neq \tau_2(i)\}$. To avoid the labeling issue, we use $\inf_{\sigma\in\mathcal{S}_K} d_H\{\sigma \circ \tau(\cdot; \mathbf{X}), \tau(\cdot; \mathbf{X}_0)\}$ as the measurement for clustering performance, where $\mathcal{S}_K$ is the set of all permutations in $[K]$.

A clustering procedure for stochastic block models is called consistent if the resulting fraction of mis-clustered vertices is asymptotically zero. Consistent clustering procedures in stochastic block models have been investigated in earlier work, including likelihood-based methods [26], spectral clustering based on the Laplacian spectral embedding [37], K-means clustering based on the adjacency spectral embedding [33], and modularity maximization [58], among others. In contrast to these frequentist approaches, our method for clustering is a Bayesian method and is straightforward via the aforementioned posterior spectral embedding: Similar to the K-means clustering based on $\widehat{\mathbf{X}}_{\mathrm{ASE}}$, we directly apply the K-means clustering procedure to realizations drawn from posterior spectral embedding. Specifically, for each realization $\mathbf{X}$ drawn from the posterior spectral embedding, we obtain a cluster assignment function $\tau(\cdot;\mathbf{X})$ by applying the aforementioned K-means clustering procedure to $\mathbf{X}$. This results in a posterior distribution of the cluster assignment function $\Pi\{\tau(\cdot;\mathbf{X})\in\cdot\mid\mathbf{A}\}$, which is induced from the map $\mathbf{X}\mapsto\tau(\cdot;\mathbf{X})$ and posterior spectral embedding $\Pi(\mathrm{d}\mathbf{X}\mid\mathbf{A})$. The below theorem shows that we can recover the clustering structure through the K-means procedure even when we assume that the working model is the random dot product graph model, which is more general than the positive semidefinite stochastic block model.

Theorem 10 *Assume the conditions in Theorem 7 hold, and let the constants $M_1, M_2 > 0$ be provided by Theorem 7. Further assume that $\mathbf{X}_0 = [\mathbf{x}_{01},\ldots,\mathbf{x}_{0n}]^{\mathrm{T}}$ has K distinct rows $\mathbf{x}_{01}^*,\ldots,\mathbf{x}_{0K}^*$ for some $K\in[n]$, they satisfy $\min_{k\neq k'}\|\mathbf{x}_{0k}^*-\mathbf{x}_{0k'}^*\|_2>\xi$ for some $\xi>0$, and $n_k := \sum_{i=1}^n \mathbb{1}(\mathbf{x}_{0i}=\mathbf{x}_{0k}^*)\to\infty$ as $n\to\infty$ for all $k\in[K]$. Then for sufficiently large n,*

$$\mathbb{E}_0\left[\Pi\left\{\inf_{\sigma\in\mathcal{S}_K} d_H(\sigma\circ\tau_0,\tau_{\mathbf{X}})\geq\frac{16M_2^2}{\xi^2}\;\Big|\;\mathbf{A}\right\}\right]\leq 8\exp\left(-\frac{1}{2}nd\right),$$

where $\tau_0=\tau(\cdot;\mathbf{X}_0)$ and $\tau_{\mathbf{X}}=\tau(\cdot;\mathbf{X})$. Let $\widehat{\mathbf{U}}$ be the left-singular vector matrix of $\widehat{\mathbf{X}}$ defined in Theorem 8, and $\mathbf{U}_0$ be that of $\mathbf{X}_0$. Then it almost always holds that

$$\inf_{\sigma\in\mathcal{S}_K} d_H\{\sigma\circ\tau(\cdot;\widehat{\mathbf{U}}),\tau(\cdot;\mathbf{U}_0)\}\leq\frac{16}{\xi^2}\left\{\frac{8M_1\sqrt{2d}}{\lambda_d(\boldsymbol{\Delta})}\right\}^2.$$

Remark 6 *There are several results related to our method in the literature in terms of clustering in stochastic block models. Strong consistency for clustering in stochastic block models was achieved in [22] and [59], but their methods were not applicable to more general random dot product graph models. In addition, their approaches are frequentist methods, whereas we take a Bayesian perspective and establish theoretical properties of the resulting full posterior distribution. A Bayesian methodology for* *clustering stochastic block models was involved in [60], but the consistency result was with regard to the maximum a posteriori estimator rather than the full posterior distribution. The strong consistency of the full posterior distribution for clustering in stochastic block models was discussed in [61], but under the assumption that the stochastic block models were homogeneous. In contrast, our work includes the positive semidefinite stochastic block models and is more flexible from the perspective of the number of free parameters.*

Remark 7 *The rate $O(1)$ for the number of mis-clustered vertices is due to the convergence rate $\mathbb{E}_0\{(1/n)\inf_{\mathbf{W}}\|\widehat{\mathbf{X}}-\mathbf{X}_0\mathbf{W}\|_{\mathrm{F}}^2\}\asymp 1/n$. This improvement is not only specific to the dot-product structure of the graph model but also accredited to the Bayesian approach, along with its specific proof strategy. The improvement is specific to dot-product structure because the minimax lower bound provided in Theorem 6 is only valid in the context of random dot product graphs. It should also be accredited to the Bayesian approach with its corresponding proof strategy because by doing so, we are able to achieve the desired minimax lower bound via Bayes estimates.*

3.5 The Gaussian spectral embedding

We have seen in Section 3.2 that the major difference between the posterior spectral embedding over the adjacency spectral embedding for the random dot product graph model is that posterior spectral embedding is a fully likelihood-based approach taking the Bernoulli likelihood information into account, while the adjacency spectral embedding only leverages the low-rank structure of the expected value of the adjacency matrix $\mathbf{X}\mathbf{X}^{\mathrm{T}}=\mathbb{E}_{\mathbf{X}}(\mathbf{A})$. Recall that the adjacency spectral embedding $\widehat{\mathbf{X}}_{\mathrm{ASE}}$ is the solution to the minimization problem

$$\min_{\mathbf{X}\in\mathbb{R}^{n\times d}}\|\mathbf{A}-\mathbf{X}\mathbf{X}^{\mathrm{T}}\|_{\mathrm{F}}^2.$$

Equivalently, we can also view $\widehat{\mathbf{X}}_{\mathrm{ASE}}$ as the maximum likelihood estimator of $\mathbf{X}$ using a Gaussian likelihood function

$$\begin{aligned}\widehat{\mathbf{X}}_{\mathrm{ASE}} &= \operatorname*{arg\,min}_{\mathbf{X}\in\mathbb{R}^{n\times d}} \|\mathbf{A} - \mathbf{X}\mathbf{X}^{\mathrm{T}}\|_{\mathrm{F}}^2 \\ &= \operatorname*{arg\,max}_{\mathbf{X}\in\mathbb{R}^{n\times d}} \sum_{i=1}^{n}\sum_{j=1}^{n}\left\{-\frac{1}{2}\log(2\pi) - \frac{1}{2}(A_{ij} - \mathbf{x}_i^{\mathrm{T}}\mathbf{x}_j)^2\right\}.\end{aligned}$$

The above interpretation motivates us to study a Bayesian version of the adjacency spectral embedding, referred to as the *Gaussian spectral embedding*, introduced as follows. Assume that Π_G is some prior distribution on the latent position matrix $\mathbf{X}$ supported on $\mathbb{R}^{n\times d}$. We consider the following pseudo-posterior distribution by taking the Gaussian density as the working model

$$\begin{aligned}&\Pi_G(\mathbf{X}\in\mathcal{A}\mid\mathbf{A}) = \frac{N_n^G(\mathcal{A})}{D_n^G},\quad N_n^G(\mathcal{A}) = \int_{\mathcal{A}}\prod_{i,j\in[n]}\frac{\phi(A_{ij} - \mathbf{x}_i^{\mathrm{T}}\mathbf{x}_j)}{\phi(A_{ij} - \mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})}\Pi_G(\mathrm{d}\mathbf{X}), \\ &\qquad D_n^G = N_n(\mathbb{R}^{n\times d}),\end{aligned} \tag{3.5}$$

for any measurable set $\mathcal{A}\subset\mathbb{R}^{n\times d}$, where ϕ is the density function of $\mathrm{N}(0,1)$. The formulation of (3.5) is completely based on the spectral property of $\mathbf{A}$ and $\mathbb{E}_{\mathbf{X}}(\mathbf{A}) = \mathbf{X}\mathbf{X}^{\mathrm{T}}$, and does not incorporate the Bernoulli likelihood information. We refer to the pseudo-posterior distribution (3.5) as the Gaussian spectral embedding of $\mathbf{A}$. Observe that when

$$\Pi_G(\mathrm{d}\mathbf{X}) = \prod_{i=1}^{n}\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^d \exp\left(-\frac{\mathbf{x}_i^{\mathrm{T}}\mathbf{x}_i}{2\sigma^2}\right)\mathrm{d}\mathbf{x}_i \tag{3.6}$$

for some $\sigma^2 > 0$, the maximum a posteriori estimator of (3.5) is the same as the solution to the minimization problem

$$\min_{\mathbf{X}\in\mathbb{R}^{n\times d}}\|\mathbf{A} - \mathbf{X}\mathbf{X}^{\mathrm{T}}\|_{\mathrm{F}}^2 + \frac{1}{2\sigma^2}\|\mathbf{X}\|_{\mathrm{F}}^2.$$

In particular, when $\sigma^2\to\infty$, which corresponds to a non-informative flat prior, the maximum a posteriori estimator of (3.5) coincides with the adjacency spectral embedding $\widehat{\mathbf{X}}_{\mathrm{ASE}}$. Therefore, one can heuristically view the Gaussian spectral embedding defined through (3.5) as a direct Bayesian analogy of the adjacency spectral embedding.

Remark 8 (Generality of the Gaussian spectral embedding) *Recall that the random dot product graph model can be alternatively regarded as a low-rank matrix model:* $\mathbf{A} = \mathbf{X}\mathbf{X}^{\mathrm{T}} + \mathbf{E}$ *for some low-rank matrix* $\mathbf{X}\mathbf{X}^{\mathrm{T}}$ *and some noise matrix* $\mathbf{E}$. *Note that in the formulation of the Gaussian spectral embedding, we do not constrain the latent positions* $\mathbf{x}_1, \ldots, \mathbf{x}_n$ *to lie in the space* $\mathcal{X} = \{\mathbf{x} \in \mathbb{R}^d : \|\mathbf{x}\|_2 < 1, \mathbf{x} > \mathbf{0}\}$, *and do not assume a parametric form for the distribution of the entries of* $\mathbf{A}$. *Namely, the Gaussian spectral embedding* (3.5) *is well-defined not only for the random dot product graph model but also for a more general class of low-rank matrix models. In the theoretical analysis below, we shall assume that the sampling model for* $\mathbf{A}$ *is a more general low-rank matrix model* $\mathbf{A} = \mathbf{X}\mathbf{X}^{\mathrm{T}} + \mathbf{E}$ *for some* $\mathbf{X} \in \mathbb{R}^{n\times d}$, *and the entries of* $\mathbf{E}$ *are only required to be sub-Gaussian.*

Theorem 11 *Let* $\mathbf{A} \in \mathbb{R}^{n\times n}$ *be a symmetric random matrix with* $(A_{ij} : 1 \leq i \leq j \leq n)$ *being independent, and let* $\mathbb{E}_0(\mathbf{A}) = \mathbf{X}_0\mathbf{X}_0^{\mathrm{T}}$ *for some* $\mathbf{X}_0 \in \mathbb{R}^{n\times d}$, *where* $d/n \to 0$. *Assume that* $(1/n)\mathbf{X}_0^{\mathrm{T}}\mathbf{X}_0 \to \boldsymbol{\Delta}$ *as* $n \to \infty$ *for some positive definite* $\boldsymbol{\Delta} \in \mathbb{R}^{d\times d}$, *and the entries of* $\mathbf{A} - \mathbb{E}_0(\mathbf{A})$ *are sub-Gaussian,* i.e., *there exists some constant* $\tau > 0$, *such that for all* $\mathbf{A} \in \mathbb{R}^{n\times n}$ *with* $\|\mathbf{A}\|_{\mathrm{F}}^2 = 1$, *and all* $t > 0$, $\mathbb{P}_0\left[\left|\mathrm{Tr}\left\{\mathbf{A}^{\mathrm{T}}(\mathbf{A} - \mathbf{X}_0\mathbf{X}_0^{\mathrm{T}})\right\}\right| > t\right] \leq \mathrm{e}^{-\tau t^2}$. *Then there exist some* $M > 0$ *and a constant* C_τ *only depending on* τ *and* $\boldsymbol{\Delta}$, *such that for sufficiently large* n,

$$\mathbb{E}\left\{\Pi_G\left(\frac{1}{n}\inf_{\mathbf{W}\in\mathbb{O}(d)}\|\mathbf{X} - \mathbf{X}_0\mathbf{W}\|_{\mathrm{F}}^2 > \frac{Md\log n}{n} \;\middle|\; \mathbf{A}\right)\right\} \leq 14\exp(-C_\tau M^2 n\log n).$$

On the one hand, when the sampling model is restricted to the random dot product graph model, the posterior contraction rate of the Gaussian spectral embedding is slower than the optimal rate $1/n$ by an extra logarithmic factor, while the posterior spectral embedding yields a rate-optimal contraction. On the other hand, the Gaussian spectral embedding can be applied to more general low-rank matrix models, whereas the posterior spectral embedding is specifically designed for the random dot product graph model. In addition, the posterior spectral embedding requires the latent positions $\mathbf{x}_1, \ldots, \mathbf{x}_n$ to lie in the space $\mathcal{X}$. Such a restriction could potentially lead to a cumbersome Markov chain Monte Carlo sampler for posterior inference. In contrast, the Gaussian spectral embedding has no constraint on the latent positions, making the corresponding posterior computation relatively convenient.

3.6 Generalization to sparse and directed graph model

We have been so far considered dense graph models where the expected number of edges is quadratic in the number of vertices. We have also restricted ourselves to the case where the edge probability matrix $\mathbf{P} = \mathbf{X}\mathbf{X}^{\mathrm{T}}$ is positive semidefinite. In this section, we aim to slightly generalize the previously obtained theoretical results to the case of sparse and directed graphs, in which the edge probability matrix is no longer required to be positive semidefinite.

We first generalize the definition of the random dot product graph model to sparse and directed graphs by introducing two latent positions for each vertex and an additional sparsity factor $\vartheta_n \in (0, 1]$. Formally, let $\mathbf{X} = [\mathbf{x}_1, \ldots, \mathbf{x}_n]^{\mathrm{T}}$, $\widetilde{\mathbf{X}} = [\widetilde{\mathbf{x}}_1, \ldots, \widetilde{\mathbf{x}}_n]^{\mathrm{T}}$ be two latent position matrices, $\mathbf{x}_i, \widetilde{\mathbf{x}}_i \in \mathcal{X}$, $i = 1, \ldots, n$, and $(\vartheta_n)_{n=1}^{\infty}$ be a sequence, referred to as the sparsity factor, such that $0 < \theta_n \leq 1$ for all n. We say that a random binary matrix $\mathbf{A} = (A_{ij})_{n\times n}$ is the adjacency matrix of a (directed) random dot product graph with latent position matrices $\mathbf{X}, \widetilde{\mathbf{X}}$ and a sparsity factor ϑ_n, denoted by $\mathbf{A} \sim \mathrm{RDPG}(\mathbf{X}, \widetilde{\mathbf{X}}; \vartheta_n)$ if $A_{ij} \sim \mathrm{Bernoulli}(\vartheta_n \mathbf{x}_i^{\mathrm{T}} \widetilde{\mathbf{x}}_j)$ for all $i, j \in [n]$. It follows that the distribution of $\mathbf{A}$ becomes

$$p(\mathbf{A} \mid \mathbf{X}, \widetilde{\mathbf{X}}) = \prod_{i,j\in[n]} (\vartheta_n \mathbf{x}_i^{\mathrm{T}} \widetilde{\mathbf{x}}_j)^{A_{ij}} (1 - \vartheta_n \mathbf{x}_i^{\mathrm{T}} \widetilde{\mathbf{x}}_j)^{1-A_{ij}}.$$

The sparsity factor ϑ_n determines whether the resulting graph is sparse or dense. If $\liminf_{n\to\infty} \vartheta_n > 0$, then the graph is dense, in the sense that $\mathbb{E}(\sum_{i,j\in[n]} A_{ij}) = O(n^2)$. If $\vartheta_n \to 0$ as $n \to \infty$, the resulting graph is sparse, *i.e.*, $\mathbb{E}(\sum_{i,j\in[n]} A_{ij}) = o(n^2)$.

Furthermore, by introducing another latent position matrix $\widetilde{\mathbf{X}}$, we relax the assumption that the edge probability matrix $\mathbb{E}(\mathbf{A})$ is positive semidefinite, and substitute it with a weaker requirement: the edge probability matrix is only required to yield a factorization of the form $\mathbb{E}(\mathbf{A}) = \mathbf{X}\widetilde{\mathbf{X}}^{\mathrm{T}}$ for some $\mathbf{X}, \widetilde{\mathbf{X}} \in \mathcal{X}^n$. Namely, we only require that the edge probability has a dot-product form. Such relaxation can also be useful to relax the positive semidefinite requirement for the block probability matrix $\mathbf{B}$ in the stochastic block model. In fact, the block probability matrix is only required to yield a dot-product form $\mathbf{B} = \mathbf{X}^*(\widetilde{\mathbf{X}}^*)^{\mathrm{T}}$ for some $\mathbf{X}^*, \widetilde{\mathbf{X}}^* \in \mathcal{X}^K$ as well .

Example (Directed stochastic block model) *A random adjacency matrix* $\mathbf{A} = [A_{ij}]_{n\times n}$ *is drawn from a (directed) stochastic block model with a block probability matrix* $\mathbf{B} = [B_{kl}]_{K\times K} \in [0,1]^{K\times K}$*, a block assignment function* $\tau : [n] \to [K]$*, and a sparsity factor* $\vartheta_n \in (0,1]$*, denoted by* $\mathbf{A} \sim \mathrm{SBM}(\mathbf{B}, \tau; \vartheta_n)$*, if the random variables* $A_{ij} \sim \mathrm{Bernoulli}(\vartheta_n B_{\tau(i)\tau(j)})$ *independently for* $i, j \in [n]$*. Assume that* $\mathrm{rank}(\mathbf{B}) = d \leq K$*. Now we do not assume that* $\mathbf{B}$ *is positive semidefinite, but only assume that it has a weaker dot-product form in the following sense: There exist two matrices* $\mathbf{X}^* = [\mathbf{x}_1^*, \ldots, \mathbf{x}_K^*]^{\mathrm{T}}, \widetilde{\mathbf{X}}^* = [\widetilde{\mathbf{x}}_1^*, \ldots, \widetilde{\mathbf{x}}_K^*]^{\mathrm{T}} \in \mathcal{X}^K$ *such that* $\mathbf{B} = \mathbf{X}^*(\widetilde{\mathbf{X}}^*)^{\mathrm{T}}$*. It follows that* $A_{ij} \sim \mathrm{Bernoulli}\left\{\vartheta_n (\mathbf{x}_{\tau(i)}^*)^{\mathrm{T}}(\widetilde{\mathbf{x}}_{\tau(j)}^*)\right\}$*. By converting the block assignment function* τ *into an* $n \times K$ *matrix* $\mathbf{Z} = [\mathbb{1}\{\tau(i) = k\}]_{i\in[n],k\in[K]}$*, we see that* $\mathrm{SBM}(\mathbf{B}, \tau; \vartheta_n)$ *coincides with* $\mathrm{RDPG}(\mathbf{X}, \widetilde{\mathbf{X}}; \vartheta_n)$ *through the reparametrization* $\mathbf{X} = \mathbf{Z}\mathbf{X}^*$ *and* $\widetilde{\mathbf{X}} = \mathbf{Z}\widetilde{\mathbf{X}}^*$*, provided that the block probability matrix* $\mathbf{B}$ *has a dot-product structure.*

We assume that the sparsity factor ϑ_n is known and assign the following prior distribution to the latent position matrices $\mathbf{X}$ and $\widetilde{\mathbf{X}}$: $\mathbf{x}_1, \ldots, \mathbf{x}_n, \widetilde{\mathbf{x}}_1, \ldots, \widetilde{\mathbf{x}}_n$ follow a distribution with a density $\pi_{\mathbf{x}}$ independently, where $\pi_{\mathbf{x}}$ is supported on $\mathcal{X}$ and is bounded away from 0 and ∞ over $\mathcal{X}$. The following posterior contraction result under the aforementioned sparse and directed Bayesian random dot product graph model holds:

Theorem 12 *Let* $\mathbf{A} \sim \mathrm{RDPG}(\mathbf{X}_0, \widetilde{\mathbf{X}}_0; \vartheta_n)$ *for some sparse factor* $(\vartheta_n)_{n=1}^{\infty}$*, latent positions matrices* $\mathbf{X}_0 = [\mathbf{x}_{01}, \ldots, \mathbf{x}_{0n}]^{\mathrm{T}}, \widetilde{\mathbf{X}}_0 = [\widetilde{\mathbf{x}}_{01}, \ldots, \widetilde{\mathbf{x}}_{0n}]^{\mathrm{T}} \in \mathbb{R}^{n\times d}$*, and the prior* Π *be described as above. If* d *is fixed (*i.e.*,* d *does not change with* n*), and* $\delta \leq \min_{i,j} \mathbf{x}_{0i}^{\mathrm{T}}\widetilde{\mathbf{x}}_{0j} \leq \max_{i,j} \mathbf{x}_{0i}^{\mathrm{T}}\widetilde{\mathbf{x}}_{0j} \leq 1-\delta$ *for some constant* $\delta \in (0, 1/2)$ *independent of* n*, then there exist some large constants* $M > 0$ *only depending on* δ*,* $\pi_{\mathbf{x}}$*, and* d*, such that*

$$\mathbb{E}_0\left\{\Pi\left(\frac{1}{n}\|\mathbf{X}\widetilde{\mathbf{X}}^{\mathrm{T}} - \mathbf{X}_0\widetilde{\mathbf{X}}_0^{\mathrm{T}}\|_{\mathrm{F}} > \frac{M}{\vartheta_n}\sqrt{\frac{d\log n}{n}} \;\middle|\; \mathbf{A}\right)\right\} \leq 3\exp\left(-\frac{1}{2}nd\log n\right)$$

for sufficiently large n*, provided that* $\vartheta_n^{-1}\sqrt{n^{-1}d\log n} \to 0$ *as* $n \to \infty$*.*

Furthermore, for the directed stochastic block model, the K-means procedure applied to the posterior spectral embedding yields the following bound on the number of mis-clustered vertices:

Theorem 13 *Assume the conditions in Theorem 12 hold. Further assume that* $\mathbf{X}_0 = [\mathbf{x}_{01}, \ldots, \mathbf{x}_{0n}]^{\mathrm{T}}, \widetilde{\mathbf{X}}_0 = [\widetilde{\mathbf{x}}_{01}, \ldots, \widetilde{\mathbf{x}}_{0n}]^{\mathrm{T}}$ *have* K *distinct rows* $\mathbf{x}_{01}^*, \ldots, \mathbf{x}_{0K}^*$*, and* $\widetilde{\mathbf{x}}_{01}^*, \ldots, \widetilde{\mathbf{x}}_{0K}^*$*, for some* $K \in [n]$*, they satisfy* $\min_{k\neq k'}\|\mathbf{x}_{0k}^* - \mathbf{x}_{0k'}^*\|_2 > \xi$*,* $\min_{k\neq k'}\|\widetilde{\mathbf{x}}_{0k}^* - \widetilde{\mathbf{x}}_{0k'}^*\|_2 > \xi$ *for some* $\xi > 0$*,* $n_k := \sum_{i=1}^n \mathbb{1}(\mathbf{x}_{0i} = \mathbf{x}_{0k}^*) \to \infty$*,* $n_k/n \to w_k$ *as* $n \to \infty$ *for all* $k \in [K]$ *for some vector* $\mathbf{w} = [w_1, \ldots, w_K]^{\mathrm{T}}$ *with* $\sum_{k=1}^K w_k = 1$*, and* $\mathbb{1}(\mathbf{x}_{0i} = \mathbf{x}_{0k}^*) = \mathbb{1}(\widetilde{\mathbf{x}}_{0i} = \widetilde{\mathbf{x}}_{0k}^*)$ *for all* $i \in [n], k \in [K]$*. Let* $\mathbf{U}, \mathbf{U}_0 \in \mathbb{O}(n, d)$ *be the left-singular vector matrices of* $\mathbf{X}\widetilde{\mathbf{X}}^{\mathrm{T}}$ *and* $\mathbf{X}_0\widetilde{\mathbf{X}}_0^{\mathrm{T}}$*, respectively. Then there exists some large constant* $M > 0$*, such that for sufficiently large* n*,*

$$\mathbb{E}_0\left\{\Pi\left[\inf_{\sigma\in\mathcal{S}_K} d_H\{\sigma\circ\tau(\cdot;\mathbf{U}_0), \tau(\cdot;\mathbf{U})\} \geq \frac{M\log n}{\vartheta_n\xi^2} \;\middle|\; \mathbf{A}\right]\right\} \leq 3\exp\left(-\frac{1}{2}nd\right).$$

3.7 Numerical examples

3.7.1 General setup for the posterior inference

We evaluate the performance of the proposed posterior spectral embedding in comparison with the Gaussian/adjacency spectral embedding through simulated examples. For each of the numerical setup, the posterior inferences are carried out through the Metropolis-Hastings sampler presented in 3.3 with 15000 iterations, where the first 5000 iterations are discarded as burn-in, and 1000 post-burn-in samples are collected every 10 iterations. The prior density for $\mathbf{x}_1, \ldots, \mathbf{x}_n$ is set to be the uniform distribution $\mathrm{Unif}(\mathcal{X})$ for the posterior spectral embedding, and the Gaussian prior in (3.6) with $\sigma = 10$ for the Gaussian spectral embedding.

3.7.2 Stochastic block models

We first consider stochastic block models with positive semidefinite block probability matrices as our simulated examples. Three simulation setups are considered, and the number of communities K and the unique values of their latent positions $[\mathbf{x}_{01}^*, \ldots, \mathbf{x}_{0K}^*]$ are tabulated in Table 3-I. In each simulation setup, the numbers of vertices in different clusters are drawn from a multinomial distribution with the probability vector $[1/K, \ldots, 1/K]^{\mathrm{T}}$.

Table 3-I. Simulation setup for positive semidefinite stochastic block models

K	$K = 3$	$K = 5$
n	$n = 600$	$n = 1000$
$[\mathbf{x}_{01}^*, \ldots, \mathbf{x}_{0K}^*]$	$\begin{bmatrix} 0.3 & 0.3 & 0.6 \\ 0.3 & 0.6 & 0.3 \end{bmatrix}$	$\begin{bmatrix} 0.3 & 0.3 & 0.7 & 0.7 & 0.5 \\ 0.3 & 0.7 & 0.3 & 0.7 & 0.5 \end{bmatrix}$
K	$K = 7$	
n	$n = 1400$	
$[\mathbf{x}_{01}^*, \ldots, \mathbf{x}_{0K}^*]$	$\begin{bmatrix} 0.2 & 0.2 & 0.2 & 0.5 & 0.5 & 0.5 & 0.7 \\ 0.2 & 0.5 & 0.7 & 0.2 & 0.5 & 0.7 & 0.2 \end{bmatrix}$	

For the posterior spectral embedding, we compute the point estimator $\widehat{\mathbf{X}}$ given in Theorem 8. A point estimator for the Gaussian spectral embedding is also obtained in a similar fashion. Note that although the data generating models are positive semidefinite stochastic block models, the posterior inferences are performed without assuming any community/cluster structure of vertices in the graph model. We perform the subsequent clustering based on the K-means procedure, as described in Section 3.4.

Table 3-II. Stochastic block models: Rand indices of different clustering methods. PSE, the posterior spectral embedding; ASE, the adjacency spectral embedding; GSE, the Gaussian spectral embedding.

Method	PSE (Point estimate)	ASE	GSE (Point estimate)
$K = 3$, $n = 600$	**0.9171**	0.9160	0.7826
$K = 5$, $n = 1000$	**0.9584**	0.9558	0.7187
$K = 7$, $n = 1400$	**0.9964**	0.9508	0.9505

The authors of [62] suggested using the Rand index to evaluate the performance of clustering. Specifically, given two partitions $\mathcal{C}_1 = \{c_{11}, \ldots, c_{1r}\}$ and $\mathcal{C}_2 = \{c_{21}, \ldots, c_{2s}\}$ of $[n]$, *i.e.*, for $i = 1, 2$, c_{ij}'s are disjoint, and their union is $[n]$, denote a the number of pairs of elements in $[n]$ that are both in the same set in $\mathcal{C}_1$ and in the same set in $\mathcal{C}_2$, and b the number of pairs in $[n]$ that are neither in the same set in $\mathcal{C}_1$ nor in the same set in $\mathcal{C}_2$. Then the Rand index RI is defined as $\mathrm{RI} = 2(a + b)/\{n(n - 1)\}$. The Rand index is a quantity between 0 and 1, with a higher value suggesting better accordance between the two partitions. In particular, when $\mathcal{C}_1$ and $\mathcal{C}_2$ are identical up to relabeling, $\mathrm{RI} = 1$.

The comparisons of the Rand indices and the embedding errors $(1/n)\inf_{\mathbf{W}} \|\widehat{\mathbf{X}} - \mathbf{X}_0\mathbf{W}\|_{\mathrm{F}}^2$ for the three embedding approaches are tabulated in Table 3-II and Table 3-III, respectively. We see that the point estimates of the posterior spectral embedding are superior to the other two competitors in terms of higher Rand indices and lower embedding errors, whereas the point estimates of the Gaussian spectral embedding perform the worst in all three setups. All three embedding approaches perform better as the number of vertices n increases. In particular, the Gaussian spectral embedding does not produce satisfactory results when $n = 600$ and $n = 1000$ but performs decently well when $n = 1400$. These numerical results are also in accordance with the theoretical results established in Sections 3.2, 3.4, and 3.5.

We also visualize the three embeddings of the observed adjacency matrix for the three setups in Figures 3-1, 3-2, and 3-3, respectively. The estimation errors of the

Table 3-III. Stochastic block models: Errors $(1/n)\inf_{\mathbf{W}}\|\widehat{\mathbf{X}}-\mathbf{X}\mathbf{W}\|_{\mathrm{F}}^2$ of different methods. PSE, the posterior spectral embedding; ASE, the adjacency spectral embedding; GSE, the Gaussian spectral embedding.

Method	PSE (Point estimate)	ASE	GSE (Point estimate)
$K=3$, $n=600$	$\mathbf{1.281\times 10^{-2}}$	1.560×10^{-2}	2.792×10^{-2}
$K=5$, $n=1000$	$\mathbf{6.851\times 10^{-3}}$	8.548×10^{-3}	1.418×10^{-2}
$K=7$, $n=1400$	$\mathbf{3.460\times 10^{-3}}$	3.582×10^{-3}	4.200×10^{-3}

point estimates under the Gaussian spectral embedding can be clearly recognized from the figures when $n=600$ and $n=1000$.

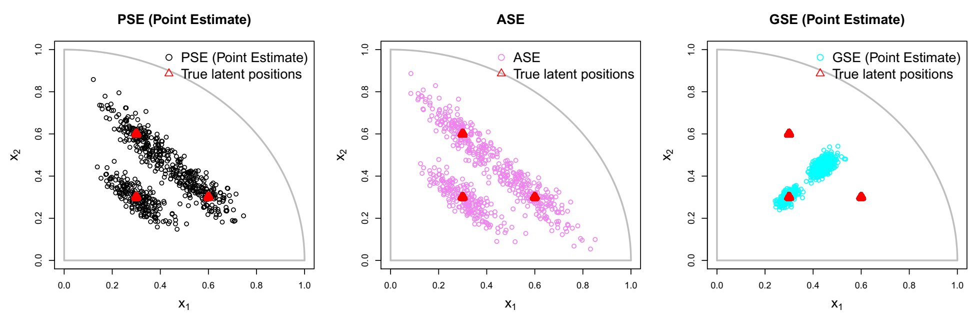


Figure 3-1. Visualization of the three embedding approaches for the simulated positive semidefinite stochastic block models with $K=3$; The red triangles are the true latent positions, and the scatter points are embedding estimates of the latent positions.

3.7.3 A Hardy-Weinberg curve example

We provide another simulation example that is different from the stochastic block model considered in Section 3.7.2. We consider the Hardy-Weinberg curve example mentioned in Chapter 1 with the latent position matrix $\mathbf{X}_0=[\mathbf{x}_{01},\ldots,\mathbf{x}_{0n}]^{\mathrm{T}}\in\mathbb{R}^{n\times d}$, where $n=2000$ and $d=3$. The latent positions $\mathbf{x}_{0i}$'s are drawn from the Hardy-Weinberg curve as follows: $\mathbf{x}_{0i}=[t_i^2, 1-2t_i+t_i^2, 2t_i-2t_i^2]^{\mathrm{T}}\in\mathbb{R}^3$, where $t_1,\ldots,t_n$ are independently drawn from $\mathrm{Unif}(0,1)$. We plot the embeddings of the observed adjacency matrix under the three approaches in panels (b), (c), and (d) of Figure 3-4, showing that the point estimates of the posterior spectral embedding produce embeddings that are closer to their true values than the other two competitors

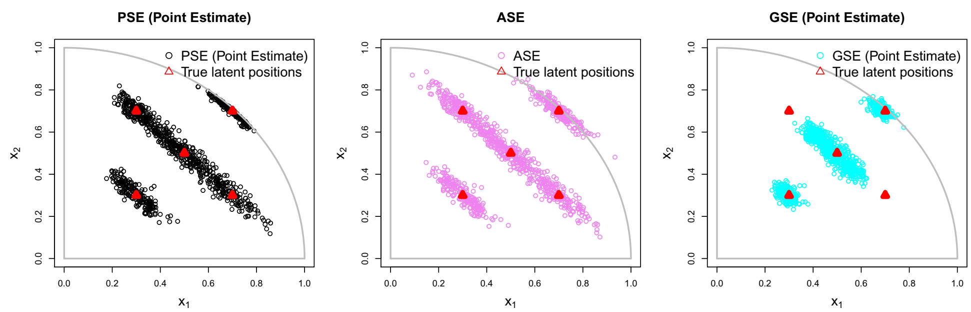


Figure 3-2. Visualization of the three embedding approaches for the simulated positive semidefinite stochastic block models with $K = 5$; The red triangles are the true latent positions, and the scatter points are embedding estimates of the latent positions.

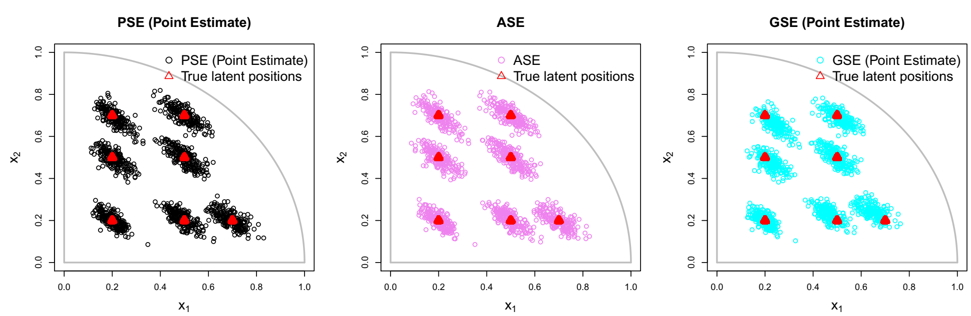


Figure 3-3. Visualization of the three embedding approaches for the simulated positive semidefinite stochastic block models with $K = 7$; The red triangles are the true latent positions, and the scatter points are embedding estimates of the latent positions.

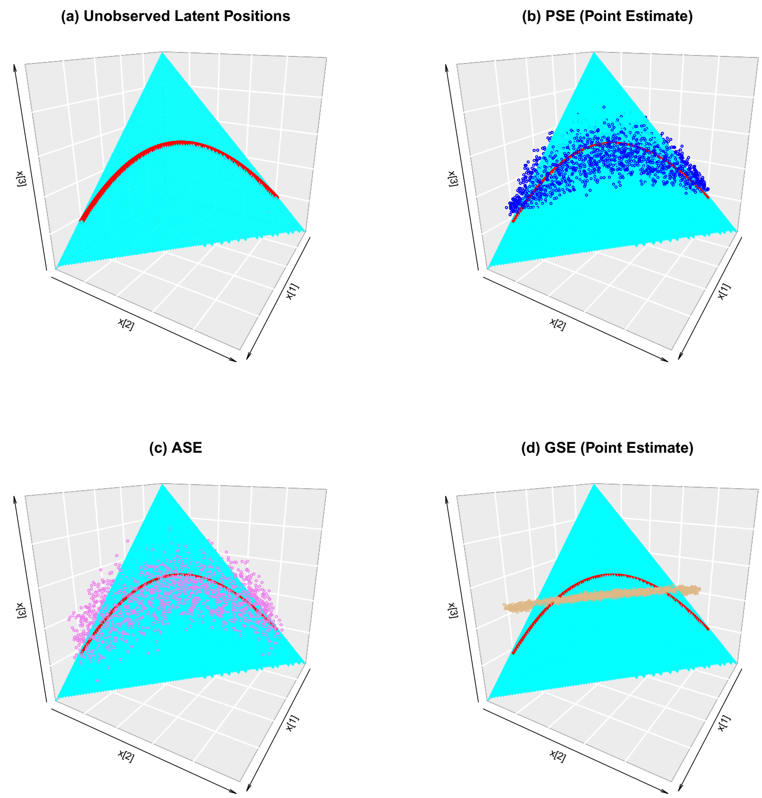


Figure 3-4. The Hardy-Weinberg curve example: The scatter points are embedding estimates of the latent positions using the point estimates of the posterior spectral embedding, the adjacency spectral embedding, and the point estimates of the Gaussian spectral embedding, respectively, and the red curve is the underlying unobserved Hardy-Weinberg curve $C(t)$ with $t \in [0, 1]$.

do. In particular, the point estimates of the Gaussian spectral embedding are not able to capture the shape of the Hardy-Weinberg curve. The embedding errors $(1/n) \inf_{\mathbf{W}} \|\widehat{\mathbf{X}} - \mathbf{X}\mathbf{W}\|_{\mathrm{F}}^2$ for the three embedding approaches are also presented in Table 3-IV, which shows that the posterior spectral embedding is superior to the adjacency and Gaussian spectral embedding empirically.

3.8 Discussion

There are several potential extensions of the proposed methodology and the corresponding theory. Firstly, the framework we have considered so far is based on the fact

Table 3-IV. Hardy-Weinberg curve example: Errors $(1/n)\inf_{\mathbf{W}}\|\widehat{\mathbf{X}}-\mathbf{X}_0\mathbf{W}\|_{\mathrm{F}}^2$ of different methods. PSE, the posterior spectral embedding; ASE, the adjacency spectral embedding; GSE, the Gaussian spectral embedding.

Method	PSE (Point estimate)	ASE	GSE (Point estimate)
Loss functions	9.148×10^{-3}	1.603×10^{-2}	1.462×10^{-2}

that the entries of the observed adjacency matrix of the network are Bernoulli random variables, *i.e.*, an unweighted network. It is also common to encounter weighted network data in a wide range of applications [63–65]. Our theory and method can be easily extended to weighted adjacency matrices, the elements of which typically follow distributions of more general forms or can be even nonparametric. Alternatively, the Gaussian spectral embedding proposed in Section 3.5 can be applied when the elements of the weighted adjacency matrix are sub-Gaussian random variables after centering. Last but not least, we assume that the embedding dimension d is known for the ease of the mathematical analysis. When d is unknown, we can first estimate d by some consistent estimator $\widehat{d}$ [33], and then perform the posterior/Gaussian spectral embedding based on $\widehat{d}$. Another flexible strategy is fully Bayesian: We can assign a prior distribution on d and let the posterior distribution adaptively select the correct dimension with moderate uncertainty. The challenge, nevertheless, is that it is non-trivial to design a reversible-jump sampler to address the cross-dimensional Monte Carlo problem for the random dot product graph model. We defer the computational issue with random d to future work. In contrast to Markov chain Monte Carlo samplers, which becomes computationally expensive when the number of vertices grows large, it is also worthwhile to develop variational Bayes methods along with the corresponding theory for random graph models.

Chapter 4

Random dot product graphs: Efficient local estimation

4.1 Overview and motivation

As introduced in Chapters 1 and 3, low-rank matrix factorization methods, or more precisely, spectral-based methods, have been widely used for estimating the latent positions for random dot product graphs due to the low expected rank as well as the signal-plus-noise structure of the observed graph adjacency matrix. Recall that in [32], the authors estimated the latent positions by solving the least-squares problem

$$\widehat{\mathbf{X}}^{(\mathrm{ASE})} = \underset{\mathbf{X}\in\mathbb{R}^{n\times d}}{\arg\min}\, \|\mathbf{A} - \mathbf{X}\mathbf{X}^{\mathrm{T}}\|_{\mathrm{F}}^2,$$

and the resulting estimator is referred to as the adjacency spectral embedding (ASE). The theory for the ASE has been well developed, as discussed in Chapter 1. Another popular spectral-based method is the *Laplacian spectral embedding* (LSE), which is defined by taking the scaled eigenvectors of the normalized Laplacian matrix of the adjacency matrix associated with the top d-largest eigenvalues [37]. The asymptotic theory of the LSE has also been established [38, 56]. In particular, the authors of [56] showed that each row of the LSE converges to a mean-zero multivariate normal distribution after proper scaling and centering, up to an orthogonal transformation.

Despite the great success of the spectral-based estimators (*i.e.*, the ASE and the LSE) for random dot product graphs, it has been pointed out in Chapter 3 that they are formulated in the low-rank matrix factorization fashion, but ignore the Bernoulli likelihood information contained in the sampling model. Therein, the minimax estimation for random dot product graphs has been successfully addressed via a Bayesian approach. However, a fundamental question remains open: is it possible that the ASE is sub-optimal for estimating the latent positions due to the negligence of the likelihood information? In this chapter, we will tackle this question by proposing a cleverly-designed likelihood-based estimator that dominates the ASE. Specifically, we first prove the sub-optimality of the ASE by showing that the asymptotic covariance matrix of each row of the ASE is inefficient. Then we propose a novel one-step procedure for estimating the latent positions and show that for each fixed vertex, the corresponding row of the proposed one-step estimator converges to a multivariate normal distribution after $\sqrt{n}$-scaling and centering at the underlying true latent position, up to an orthogonal transformation. More importantly, the corresponding asymptotic covariance matrix is as efficient as the maximum likelihood estimator as if the rest of the latent positions are known, provided that the procedure is initialized at an estimator satisfying the so-called approximate linearization property, which will be defined later. In particular, we show that the efficient covariance matrix is no greater than the asymptotic covariance matrix of the corresponding row of the ASE in spectra. Besides the local efficiency for each fixed vertex, the proposed one-step estimator for the latent positions has a smaller sum of squared errors (*i.e.*, the global loss function $L(\cdot,\cdot)$ defined in Section 3.1 up to an n-scaling) than the ASE globally for all vertices as well.

The general one-step procedure, which finds a new estimator via a single iteration of the Newton-Raphson update given a $\sqrt{n}$-consistent initial estimator, has been applied to M-estimation theory in classical parametric models to produce an efficient estimator [66]. Even when the maximum likelihood estimator does not exist (*e.g.*, Gaussian mixture models), the one-step procedure could still be efficient. This motivates us to extend the one-step procedure from classical parametric models to efficient estimation in high-dimensional random graphs because neither the existence nor the uniqueness of the maximum likelihood estimator for random dot product graphs has been established. Unlike the ASE, the proposed one-step procedure takes both the low-rank structure of the mean matrix and the likelihood information of the sampling model into account simultaneously. This chapter, which corresponds to the preprint [3], represents, to the best of our knowledge, the first effort in the literature addressing the efficient estimation problem for random dot product graphs.

Moreover, we prove the asymptotic sub-optimality of the widely adopted LSE by applying the one-step procedure to construct an estimator for the population version of the LSE and show that it dominates the sample version of the LSE in the following sense: Locally for each fixed vertex, the corresponding row of the new estimator converges to a mean-zero multivariate normal distribution after proper scaling and centering, up to an orthogonal transformation, with the asymptotic covariance matrix no greater than that of the corresponding row of the (sample) LSE in spectra; Globally for all vertices, it yields a smaller sum of squared error than the (sample) LSE.

The structure of the rest of this chapter is presented as follows. We begin with the necessary preliminaries for random dot product graphs as well as the ASE in Section 4.2, including the distributional result of the ASE obtained originally in [36], as well as the theory of local efficiency in the context of random dot product graphs. Section 4.3 elaborates on the proposed novel one-step procedure for estimating the entire latent position matrix, establishes its asymptotic theory, and shows that it dominates the ASE as the number of vertices goes to infinity. In Section 4.4, we apply the proposed one-step procedure to construct an estimator for the population LSE and show that it dominates the sample LSE asymptotically. Section 4.5 demonstrates the usefulness of the proposed one-step procedure via numerical examples and the analysis of a real-world Wikipedia graph data. We conclude the chapter with a discussion in Section 4.6.

4.2 Preliminaries

This section elaborates on certain preliminaries that are useful throughout the chapter. In Section 3.6, we have introduced the generalization of random dot product graphs to sparse and directed networks. In this chapter, we shall make a similar extension of the graph model to a more general scenario that allows the occurence of both sparse and dense but undirected graphs. Still denote $\mathcal{X} = \{\mathbf{x} = [x_1, \ldots, x_d]^{\mathrm{T}} \in \mathbb{R}^d : x_1, \ldots, x_d > 0, \|\mathbf{x}\| < 1\}$ the space of latent positions as in Chapter 3, and $\mathcal{X}^n = \{\mathbf{X} = [\mathbf{x}_1, \ldots, \mathbf{x}_n]^{\mathrm{T}} \in \mathbb{R}^{n\times d} : \mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathcal{X}\}$. In thic chapter the sparsity factor is denoted by $\rho_n \in (0, 1]$, and given an $n \times d$ matrix $\mathbf{X} = [\mathbf{x}_1, \ldots, \mathbf{x}_n]^{\mathrm{T}} \in \mathcal{X}^n$, a symmetric and hollow (*i.e.*, the diagonal entries are zeros) random matrix $\mathbf{A} = [A_{ij}]_{n\times n} \in \{0, 1\}^{n\times n}$ is said to be the adjacency matrix of a random dot product graph on n vertices $[n] = \{1, 2, \ldots, n\}$ with latent positions $\mathbf{X}$, denoted by $\mathbf{A} \sim \mathrm{RDPG}(\mathbf{X})$, if $A_{ij} \sim \mathrm{Bernoulli}(\rho_n \mathbf{x}_i^{\mathrm{T}} \mathbf{x}_j)$ independently, $1 \le i < j \le n$. Namely, the distribution of $\mathbf{A}$ can be written as

$$p_{\mathbf{X}}(\mathbf{A}) = \prod_{i<j} (\rho_n \mathbf{x}_i^{\mathrm{T}} \mathbf{x}_j)^{A_{ij}} (1 - \rho_n \mathbf{x}_i^{\mathrm{T}} \mathbf{x}_j)^{1-A_{ij}}.$$

Although we do not require the adjacency matrix to be hollow in Chapter 3, it turns out that whether or not allowing the occurrence of self-loops in the resulting graphs does not affect the corresponding asymptotic analysis. In this chapter, we follow the usual convention in the literature of random graph inference and eliminate the occurrence of self-loops in graphs.

The goal of this chapter is similar to Chapter 3, namely, to estimate the latent positions $\mathbf{x}_1, \ldots, \mathbf{x}_n$, but we will focus on the local behavior of an embedding estimator. Throughout this chapter, we require that there exists some cumulative distribution function F on $\mathcal{X}$, such that

$$\sup_{\mathbf{x}\in\mathcal{X}} |F_n(\mathbf{x}) - F(\mathbf{x})| \to 0 \quad \text{as } n \to \infty, \tag{4.1}$$

where $F_n(\mathbf{x}) = (1/n)\sum_{i=1}^n \mathbb{1}\{\mathbf{x}_i \leq \mathbf{x}\}$ is the empirical distribution function by treating the latent positions $\mathbf{x}_1, \ldots, \mathbf{x}_n$ as independent and identically distributed (i.i.d.) data. In some cases, the latent positions $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are considered as latent random variables that are independently sampled from some underlying distribution F on $\mathcal{X}$ (see, for example, [36, 32, 39, 56]). Condition (4.1) is similar to the case where $\mathbf{x}_i$'s are random in the following sense: When $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are independent random variables sampled from F, the Glivenko-Cantelli theorem asserts that (4.1) holds with probability one with respect to the randomness of the infinite i.i.d. sequence $(\mathbf{x}_i)_{i=1}^\infty$. We also remark that condition (4.1) is stronger than the assumption in Theorem 7, which only requires

$$\frac{1}{n}\mathbf{X}_0^{\mathrm{T}}\mathbf{X}_0 = \frac{1}{n}\sum_{i=1}^n \mathbf{x}_{0i}\mathbf{x}_{0i}^{\mathrm{T}} = \int_{\mathcal{X}} \mathbf{x}\mathbf{x}^{\mathrm{T}} F_n(\mathrm{d}\mathbf{x}) \to \mathbf{\Delta}$$

for some deterministic positive definite $\mathbf{\Delta} \in \mathbb{R}^{d\times d}$.

Recall that the authors of [32] proposed to solve the least-squares problem

$$\widehat{\mathbf{X}}^{(\mathrm{ASE})} = \underset{\mathbf{X}\in\mathbb{R}^{n\times d}}{\arg\min} \|\mathbf{A} - \mathbf{X}\mathbf{X}^{\mathrm{T}}\|_{\mathrm{F}}^2 \tag{4.2}$$

to estimate the latent positions, and the resulting solution $\widehat{\mathbf{X}}^{(\mathrm{ASE})}$ is referred to as the adjacency spectral embedding of $\mathbf{A}$ into $\mathbb{R}^d$. In the context of random dot product graphs that contain sparsity factor $(\rho_n)_{n=1}^\infty$, the authors of [56] proved that $\widehat{\mathbf{X}}^{(\mathrm{ASE})} = [\widehat{\mathbf{x}}_1^{(\mathrm{ASE})}, \ldots, \widehat{\mathbf{x}}_n^{(\mathrm{ASE})}]^{\mathrm{T}}$ is a consistent estimator for $\rho_n^{1/2}\mathbf{X}$ globally for all vertices: $(1/n)\|\widehat{\mathbf{X}}^{(\mathrm{ASE})}\mathbf{W} - \rho_n^{1/2}\mathbf{X}\|_{\mathrm{F}}^2$ converges to 0 in probability as $n \to \infty$ for some orthogonal $\mathbf{W} \in \mathbb{O}(d)$ (recall that the orthogonal alignment $\mathbf{W}$ is needed due to the intrinsic non-identifiability of the random dot product graph model with regard to orthogonal transformation of the latent positions). Furthermore, for each fixed vertex $i \in [n]$, the asymptotic distribution of $\widehat{\mathbf{x}}_i^{(\mathrm{ASE})}$ after proper scaling and centering has been established [36, 56]. Denote $\mathbf{X}_0$ the true latent position matrix that generates the observed adjacency matrix $\mathbf{A}$ according to the sampling model $\mathbf{A} \sim \mathrm{RDPG}(\mathbf{X}_0)$ and $\mathbf{X}$ any latent position matrix in $\mathcal{X}^n$. We summarize these findings in the following theorem.

Theorem 14 (Theorems 2.1 and 2.2 of [56]) *Let $\mathbf{A} \sim \mathrm{RDPG}(\mathbf{X}_0)$ with a sparsity factor ρ_n and condition (4.1) hold for some $\mathbf{X}_0 = [\mathbf{x}_{01}, \ldots, \mathbf{x}_{0n}]^{\mathrm{T}} \in \mathcal{X}^n$. Suppose either $\rho_n \equiv 1$ for all n or $\rho_n \to 0$ but $(\log n)^4/(n\rho_n) \to 0$ as $n \to \infty$, and denote $\rho = \lim_{n\to\infty} \rho_n$. Let $\widehat{\mathbf{X}}^{(\mathrm{ASE})} = [\widehat{\mathbf{x}}_1^{(\mathrm{ASE})}, \ldots, \widehat{\mathbf{x}}_n^{(\mathrm{ASE})}]^{\mathrm{T}}$ be the ASE defined by (4.2). Denote*

$$\mathbf{\Delta} = \int_{\mathcal{X}} \mathbf{x}\mathbf{x}^{\mathrm{T}} F(\mathrm{d}\mathbf{x}), \quad \text{and} \quad \mathbf{\Sigma}(\mathbf{x}) = \mathbf{\Delta}^{-1} \left[\int_{\mathcal{X}} \left\{ \mathbf{x}_1^{\mathrm{T}}\mathbf{x} \left(1 - \rho \mathbf{x}_1^{\mathrm{T}}\mathbf{x}\right) \right\} \mathbf{x}_1\mathbf{x}_1^{\mathrm{T}} F(\mathrm{d}\mathbf{x}_1) \right] \mathbf{\Delta}^{-1},$$

and assume that $\mathbf{\Delta}$ and $\mathbf{\Sigma}(\mathbf{x})$ are strictly positive definite for all $\mathbf{x}$. Then there exists an orthogonal matrix $\mathbf{W} \in \mathbb{R}^{d\times d}$ such that

$$\|\widehat{\mathbf{X}}^{(\mathrm{ASE})}\mathbf{W} - \rho_n^{1/2}\mathbf{X}_0\|_{\mathrm{F}}^2 \overset{a.s.}{\to} \int_{\mathcal{X}} \mathrm{tr}\{\mathbf{\Sigma}(\mathbf{x})\} F(\mathrm{d}\mathbf{x}), \tag{4.3}$$

and for any fixed index $i \in [n]$,

$$\sqrt{n}(\mathbf{W}^{\mathrm{T}}\widehat{\mathbf{x}}_i^{(\mathrm{ASE})} - \rho_n^{1/2}\mathbf{x}_{0i}) \overset{\mathcal{L}}{\to} \mathrm{N}(\mathbf{0}, \mathbf{\Sigma}(\mathbf{x}_{0i})). \tag{4.4}$$

Theorem 14 suggests the following two properties of the ASE: Globally for all vertices, $\widehat{\mathbf{X}}^{(\mathrm{ASE})}$ is a consistent estimator for $\rho_n^{1/2}\mathbf{X}_0$ as there exists an orthogonal matrix $\mathbf{W} \in \mathbb{R}^{d\times d}$ such that the sum of squared errors $\|\widehat{\mathbf{X}}^{(\mathrm{ASE})}\mathbf{W} - \rho_n^{1/2}\mathbf{X}_0\|_{\mathrm{F}}^2 = nL(\widehat{\mathbf{X}}^{(\mathrm{ASE})}, \rho_n^{1/2}\mathbf{X}_0)$ can be fully characterized by (4.3) as $n \to \infty$; Locally, for each fixed vertex $i \in [n]$, the distribution of the ith row $\widehat{\mathbf{x}}_i^{(\mathrm{ASE})}$ of $\widehat{\mathbf{X}}^{(\mathrm{ASE})}$ after $\sqrt{n}$-scaling and centering at $\rho_n^{1/2}\mathbf{x}_{0i}$, converges to a mean-zero multivariate normal distribution with covariance matrix $\mathbf{\Sigma}(\mathbf{x}_{0i})$, up to an orthogonal transformation $\mathbf{W}$. Nevertheless, it remains open whether the results of Theorem 14 are optimal. In Section 4.3, we will propose an estimator $\widehat{\mathbf{X}}$ that dominates the ASE asymptotically in the following sense: Globally for all vertices, it yields a smaller asymptotic sum of squared errors

$\|\widehat{\mathbf{X}}\mathbf{W} - \rho_n^{1/2}\mathbf{X}_0\|_{\mathrm{F}}^2$ than (4.3) for some orthogonal $\mathbf{W}$; Locally for each fixed vertex $i \in [n]$, the corresponding row of $\widehat{\mathbf{X}}$, after $\sqrt{n}$-scaling and centering at $\rho_n^{1/2}\mathbf{x}_{0i}$, also converges to a mean-zero multivariate normal distribution, up to an orthogonal transformation $\mathbf{W}$, but the asymptotic covariance matrix is upper bounded by $\mathbf{\Sigma}(\mathbf{x}_{0i})$ in spectra.

Before diving into the joint estimation of the entire latent position matrix $\mathbf{X}_0$, we begin with estimating a single latent position $\mathbf{x}_{0i}$ when the rest of the latent positions are known, which motivates the development of the proposed efficient estimation procedure. Specifically, for a fixed $i \in [n]$, we estimate $\mathbf{x}_{0i}$ via the maximum likelihood estimator, assuming that the rest of the latent positions $\{\mathbf{x}_{0j} : j \in [n], j \neq i\}$ are known. For simplicity, we assume that the sparsity factor $\rho_n \equiv 1$ for all n in the following theorem.

Theorem 15 *Let $\mathbf{A} \sim \mathrm{RDPG}(\mathbf{X}_0)$ for some $\mathbf{X}_0 = [\mathbf{x}_{01}, \dots, \mathbf{x}_{0n}]^{\mathrm{T}} \in \mathcal{X}^n$, and condition (4.1) hold. Assume that there exists some constant $\delta > 0$ such that $\delta \leq \min_{l,j} \mathbf{x}_{0l}^{\mathrm{T}}\mathbf{x}_{0j} \leq \max_{l,j} \mathbf{x}_{0l}^{\mathrm{T}}\mathbf{x}_{0j} \leq 1-\delta$. Fixing $i \in [n]$, we estimate $\mathbf{x}_{0i}$ with $\{\mathbf{x}_{0j} : j \in [n], j \neq i\}$ being known, and suppose the parameter space is $\Theta_n = \{\mathbf{x} \in \mathcal{X} : \delta \leq \min_{j\in[n]} \mathbf{x}^{\mathrm{T}}\mathbf{x}_{0j} \leq \max_{j\in[n]} \mathbf{x}^{\mathrm{T}}\mathbf{x}_{0j} \leq 1-\delta\}$. Further assume that $\mathbf{x}_{0i}$ is in the interior of Θ_n, and for any $\mathbf{x} \in \Theta_n$, denote*

$$\mathbf{G}(\mathbf{x}) = \int_{\mathcal{X}} \left\{ \frac{\mathbf{x}_1\mathbf{x}_1^{\mathrm{T}}}{\mathbf{x}^{\mathrm{T}}\mathbf{x}_1(1-\mathbf{x}^{\mathrm{T}}\mathbf{x}_1)} \right\} F(\mathrm{d}\mathbf{x}_1).$$

Then the maximum likelihood estimator $\widehat{\mathbf{x}}_i^{(\mathrm{MLE})} = \arg\max_{\mathbf{x}\in\Theta_n} \ell_{\mathbf{A}}(\mathbf{x})$ is consistent for $\mathbf{x}_{0i}$, and

$$\sqrt{n}(\widehat{\mathbf{x}}_i^{(\mathrm{MLE})} - \mathbf{x}_{0i}) \xrightarrow{\mathcal{L}} \mathrm{N}(\mathbf{0}, \mathbf{G}(\mathbf{x}_{0i})^{-1}). \tag{4.5}$$

Recall that for the ith row $\widehat{\mathbf{x}}_i^{(\mathrm{ASE})}$ of the ASE, $\sqrt{n}(\mathbf{W}^{\mathrm{T}}\widehat{\mathbf{x}}_i^{(\mathrm{ASE})} - \mathbf{x}_{0i}) \xrightarrow{\mathcal{L}} \mathrm{N}(\mathbf{0}, \mathbf{\Sigma}(\mathbf{x}_{0i}))$ for some orthogonal $\mathbf{W} \in \mathbb{O}(d)$ by Theorem 14. We now claim that $\mathbf{\Sigma}(\mathbf{x}_{0i}) - \mathbf{G}(\mathbf{x}_{0i})^{-1}$ is positive semidefinite. In fact, since $F_n(\cdot) = (1/n)\sum_{i=1}^n \mathbb{1}\{\mathbf{x}_i \leq \cdot\}$ converges to F in total variation distance according to condition (4.1), it follows that for any $\mathbf{x} \in \Theta_n$,

$$
\begin{aligned}
\mathbf{\Delta}_n &:= \int_{\mathcal{X}} \mathbf{x}\mathbf{x}^{\mathrm{T}} F_n(\mathrm{d}\mathbf{x}) = \frac{1}{n}\mathbf{X}_0^{\mathrm{T}}\mathbf{X}_0 \to \mathbf{\Delta}, \\
\mathbf{\Sigma}_n(\mathbf{x}) &:= \mathbf{\Delta}_n^{-1}\left[\int_{\mathcal{X}} \{\mathbf{x}^{\mathrm{T}}\mathbf{x}_1(1 - \mathbf{x}^{\mathrm{T}}\mathbf{x}_1)\}\mathbf{x}_1\mathbf{x}_1^{\mathrm{T}} F_n(\mathrm{d}\mathbf{x}_1)\right]\mathbf{\Delta}_n^{-1} \\
&= \left(\frac{1}{n}\mathbf{X}_0^{\mathrm{T}}\mathbf{X}_0\right)^{-1}\left(\frac{1}{n}\mathbf{X}_0^{\mathrm{T}}\mathbf{D}(\mathbf{x})\mathbf{X}_0\right)\left(\frac{1}{n}\mathbf{X}_0^{\mathrm{T}}\mathbf{X}_0\right)^{-1} \to \mathbf{\Sigma}(\mathbf{x}_{0i}), \\
\mathbf{G}_n(\mathbf{x}) &:= \int_{\mathcal{X}} \left\{\frac{\mathbf{x}_1\mathbf{x}_1^{\mathrm{T}}}{\mathbf{x}^{\mathrm{T}}\mathbf{x}_1(1 - \mathbf{x}^{\mathrm{T}}\mathbf{x}_1)}\right\} F_n(\mathrm{d}\mathbf{x}_1) = \frac{1}{n}\mathbf{X}_0^{\mathrm{T}}\mathbf{D}(\mathbf{x})^{-1}\mathbf{X}_0 \to \mathbf{G}(\mathbf{x}),
\end{aligned}
$$

where $\mathbf{D}(\mathbf{x}) = \mathrm{diag}\{\mathbf{x}^{\mathrm{T}}\mathbf{x}_{01}(1 - \mathbf{x}^{\mathrm{T}}\mathbf{x}_{01}), \ldots, \mathbf{x}^{\mathrm{T}}\mathbf{x}_{0n}(1 - \mathbf{x}^{\mathrm{T}}\mathbf{x}_{0n})\}$. Now let $\mathbf{X}_0$ yield the singular value decomposition $\mathbf{X}_0 = \mathbf{U}_0\mathbf{S}_0^{1/2}\mathbf{V}_0^{\mathrm{T}}$ with $\mathbf{U}_0 \in \mathbb{O}(n, d)$, $\mathbf{S}_0^{1/2}$ being diagonal, and $\mathbf{V}_0 \in \mathbb{O}(d)$. We see immediately that

$$
\begin{aligned}
\mathbf{\Sigma}_n(\mathbf{x}) &= \left(\frac{1}{n}\mathbf{X}_0^{\mathrm{T}}\mathbf{X}_0\right)^{-1}\left(\frac{1}{n}\mathbf{X}_0^{\mathrm{T}}\mathbf{D}(\mathbf{x})\mathbf{X}_0\right)\left(\frac{1}{n}\mathbf{X}_0^{\mathrm{T}}\mathbf{X}_0\right)^{-1} \\
&= n(\mathbf{V}_0\mathbf{S}_0^{-1}\mathbf{V}_0^{\mathrm{T}})(\mathbf{V}_0\mathbf{S}_0^{1/2}\mathbf{U}_0^{\mathrm{T}}\mathbf{D}(\mathbf{x})\mathbf{U}_0\mathbf{S}_0^{1/2}\mathbf{V}_0^{\mathrm{T}})(\mathbf{V}_0\mathbf{S}_0^{-1}\mathbf{V}_0^{\mathrm{T}}) \\
&= n\mathbf{V}_0\mathbf{S}_0^{-1/2}(\mathbf{U}_0^{\mathrm{T}}\mathbf{D}(\mathbf{x})\mathbf{U}_0)\mathbf{S}_0^{-1/2}\mathbf{V}_0^{\mathrm{T}}, \\
\mathbf{G}_n(\mathbf{x})^{-1} &= n(\mathbf{X}_0^{\mathrm{T}}\mathbf{D}(\mathbf{x})^{-1}\mathbf{X}_0)^{-1} = n(\mathbf{V}_0\mathbf{S}_0^{1/2}\mathbf{U}_0^{\mathrm{T}}\mathbf{D}(\mathbf{x})^{-1}\mathbf{U}_0\mathbf{S}_0^{1/2}\mathbf{V}_0^{\mathrm{T}})^{-1} \\
&= n\mathbf{V}_0\mathbf{S}_0^{-1/2}(\mathbf{U}_0^{\mathrm{T}}\mathbf{D}(\mathbf{x})^{-1}\mathbf{U}_0)^{-1}\mathbf{S}_0^{-1/2}\mathbf{V}_0^{\mathrm{T}}.
\end{aligned}
$$

Since $\mathbf{U}_0^{\mathrm{T}}\mathbf{U}_0 = \mathbf{I}_d$, it follows that $\mathbf{U}_0^{\mathrm{T}}\mathbf{D}(\mathbf{x})\mathbf{U}_0 - (\mathbf{U}_0^{\mathrm{T}}\mathbf{D}(\mathbf{x})^{-1}\mathbf{U}_0)^{-1}$ is positive semidefinite [67], and hence, $\mathbf{\Sigma}(\mathbf{x}) - \mathbf{G}(\mathbf{x})^{-1} = \lim_{n\to\infty}\{\mathbf{\Sigma}_n(\mathbf{x}_{0i}) - \mathbf{G}_n(\mathbf{x}_{0i})^{-1}\}$ is positive semidefinite for any $\mathbf{x} \in \Theta_n$. The indication of this result is that the ASE is *inefficient* for estimating the latent position $\mathbf{x}_{0i}$ for vertex i when the rest of the latent positions are known in terms of the asymptotic covariance matrix, in contrast to the efficiency of the maximum likelihood estimator. We will see in Section 4.3 that when all the latent positions are unknown, we can still construct an estimator $\widehat{\mathbf{X}} = [\widehat{\mathbf{x}}_1, \ldots, \widehat{\mathbf{x}}_n]^{\mathrm{T}}$, such that for each fixed vertex i, $\sqrt{n}(\mathbf{W}^{\mathrm{T}}\widehat{\mathbf{x}}_i - \rho_n^{1/2}\mathbf{x}_{0i})$ converges to a multivariate normal distribution up to an orthogonal $\mathbf{W} \in \mathbb{O}(d)$, but the covariance matrix is the same as that of the maximum likelihood estimator as if the rest of the latent positions are known. Correspondingly, any estimator that has such property is called a *locally efficient* estimator with regard to the orthogonal alignment $\mathbf{W}$.

4.3 The one-step estimator

The inefficiency of the ASE, indicated by $\mathbf{\Sigma}(\mathbf{x}_{0i}) \succeq \mathbf{G}(\mathbf{x}_{0i})^{-1}$, is due to the fact that the ASE is a least-squares type estimator that does not depend on the likelihood function of the sampling model. In contrast, the maximum likelihood estimator $\widehat{\mathbf{x}}_i^{(\mathrm{MLE})}$ incorporates the Bernoulli log-likelihood function $\ell_{\mathbf{A}}(\mathbf{x}) = \sum_{j\neq i}\{A_{ij}\log(\mathbf{x}^{\mathrm{T}}\mathbf{x}_{0j}) + (1-A_{ij})\log(1-\mathbf{x}^{\mathrm{T}}\mathbf{x}_{0j})\}$ and hence is asymptotically efficient. One strategy for taking advantage of the likelihood information is the maximum likelihood method for estimating the entire latent position matrix $\mathbf{X}$ as an alternative to the ASE. Unfortunately, when all latent positions are unknown, the random dot product graph model belongs to a curved exponential family rather than a canonical exponential family, and hence, neither the existence nor the uniqueness of the maximum likelihood estimator of random dot product graphs has been established. As pointed out in [68], properties of the maximum likelihood estimator in curved exponential families are harder to develop than the canonical ones. Therefore we seek another approach to find an estimator that is asymptotically equivalent to the maximum likelihood estimator. Recall that when $\{\mathbf{x}_{0j} : j \in [n], j \neq i\}$ are known, the maximum likelihood estimator is a solution to the estimating equation

$$\mathbf{\Psi}_n(\mathbf{x}) := \frac{1}{n}\sum_{j\neq i}^{n}\frac{(A_{ij}-\mathbf{x}^{\mathrm{T}}\mathbf{x}_{0j})\mathbf{x}_{0j}}{\mathbf{x}^{\mathrm{T}}\mathbf{x}_{0j}(1-\mathbf{x}^{\mathrm{T}}\mathbf{x}_{0j})} = \mathbf{0}.$$

Then given an "appropriate" initial guess of the solution $\widetilde{\mathbf{x}}_i$, we can perform a one-step Newton-Raphson update to obtain another estimator $\widehat{\mathbf{x}}_i^{(\mathrm{OS})}$ that is closer to the zero of the estimating equation $\mathbf{\Psi}_n$ (see, for example, Section 5.7 of [66]):

$$\widehat{\mathbf{x}}_i^{(\mathrm{OS})} = \widetilde{\mathbf{x}}_i + \left\{\frac{1}{n}\sum_{j\neq i}^{n}\frac{\mathbf{x}_{0j}\mathbf{x}_{0j}^{\mathrm{T}}}{\widetilde{\mathbf{x}}_i^{\mathrm{T}}\mathbf{x}_{0j}(1-\widetilde{\mathbf{x}}_i^{\mathrm{T}}\mathbf{x}_{0j})}\right\}^{-1}\left\{\frac{1}{n}\sum_{j\neq i}^{n}\frac{(A_{ij}-\widetilde{\mathbf{x}}_i^{\mathrm{T}}\mathbf{x}_{0j})\mathbf{x}_{0j}}{\widetilde{\mathbf{x}}_i^{\mathrm{T}}\mathbf{x}_{0j}(1-\widetilde{\mathbf{x}}_i^{\mathrm{T}}\mathbf{x}_{0j})}\right\}. \tag{4.6}$$

In the case of estimating $\mathbf{x}_{0i}$ with the rest of the latent positions being known, the requirement for $\widetilde{\mathbf{x}}_i$ is that it is $\sqrt{n}$-consistent for $\mathbf{x}_{0i}$, and the resulting one-step estimator $\widehat{\mathbf{x}}_i^{(\mathrm{OS})}$ is as efficient as the maximum likelihood estimator $\widehat{\mathbf{x}}_i^{(\mathrm{MLE})}$ [66].

The above result motivates us to generalize the one-step estimator (4.6) to the case where the latent positions $\mathbf{x}_{01}, \ldots, \mathbf{x}_{0n}$ are all unknown. Let $\widetilde{\mathbf{X}} = [\widetilde{\mathbf{x}}_1, \ldots, \widetilde{\mathbf{x}}_n]^{\mathrm{T}} \in \mathbb{R}^{n\times d}$ be an initial estimator $\widetilde{\mathbf{X}}$ for $\mathbf{X}_0$. An intuitive choice for generalizing the one-step updating scheme (4.6) to the case of unknown $(\mathbf{x}_{0j})_{j\neq i}$ is to substitute the unknown $\mathbf{x}_{0j}$ by the initial estimator $\widetilde{\mathbf{x}}_j$, $j \neq i$ and $j \in [n]$ in (4.6). We thus define the following one-step estimator $\widehat{\mathbf{X}} = [\widehat{\mathbf{x}}_1, \ldots, \widehat{\mathbf{x}}_n]^{\mathrm{T}}$ for $\mathbf{X}_0$:

$$\widehat{\mathbf{x}}_i = \widetilde{\mathbf{x}}_i + \left\{\frac{1}{n}\sum_{j=1}^n \frac{\widetilde{\mathbf{x}}_j\widetilde{\mathbf{x}}_j^{\mathrm{T}}}{\widetilde{\mathbf{x}}_i^{\mathrm{T}}\widetilde{\mathbf{x}}_j(1-\widetilde{\mathbf{x}}_i^{\mathrm{T}}\widetilde{\mathbf{x}}_j)}\right\}^{-1}\left\{\frac{1}{n}\sum_{j=1}^n \frac{(A_{ij}-\widetilde{\mathbf{x}}_i^{\mathrm{T}}\widetilde{\mathbf{x}}_j)\widetilde{\mathbf{x}}_j}{\widetilde{\mathbf{x}}_i^{\mathrm{T}}\widetilde{\mathbf{x}}_j(1-\widetilde{\mathbf{x}}_i^{\mathrm{T}}\widetilde{\mathbf{x}}_j)}\right\}, \quad i = 1, 2, \ldots, n. \tag{4.7}$$

Unlike the coarse $\sqrt{n}$-consistency requirement for the initial estimator in the case of estimating a single latent position with the rest being known, we need to require that the initial estimator $\widetilde{\mathbf{X}} = [\widetilde{\mathbf{x}}_1, \ldots, \widetilde{\mathbf{x}}_n]$ for the entire latent position matrix $\mathbf{X}_0$ satisfies a finer condition, referred to as the *approximate linearization property*, which is defined below.

Definition 1 (Approximate linearization property) *Given* $\mathbf{A} \sim \mathrm{RDPG}(\mathbf{X}_0)$ *with a sparsity factor* ρ_n*, where* $\mathbf{X}_0 = [\mathbf{x}_{01}, \ldots, \mathbf{x}_{0n}]^{\mathrm{T}}$*, an estimator* $\widetilde{\mathbf{X}} = [\widetilde{\mathbf{x}}_1, \ldots, \widetilde{\mathbf{x}}_n]^{\mathrm{T}}$ *is said to satisfy the approximate linearization property, if there exists an orthogonal matrix* $\mathbf{W} \in \mathbb{O}(d)$ *and an* $n \times d$ *matrix* $\widetilde{\mathbf{R}} = [\widetilde{\mathbf{R}}_1, \ldots, \widetilde{\mathbf{R}}_n]^{\mathrm{T}}$ *with* $\|\widetilde{\mathbf{R}}\|_{\mathrm{F}}^2 = O_{\mathbb{P}_0}((n\rho_n)^{-1}(\log n)^{\omega})$ *for some* $\omega \geq 0$*, such that*

$$\mathbf{W}^{\mathrm{T}}\widetilde{\mathbf{x}}_i - \rho_n^{1/2}\mathbf{x}_{0i} = \rho_n^{-1/2}\sum_{j=1}^n (A_{ij} - \rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})\boldsymbol{\zeta}_{ij} + \widetilde{\mathbf{R}}_i, \quad i = 1, 2, \ldots, n, \tag{4.8}$$

where $\{\boldsymbol{\zeta}_{ij} : i, j \in [n]\}$ *is a collection of vectors in* $\mathbb{R}^d$ *with* $\sup_{i,j\in[n]}\|\boldsymbol{\zeta}_{ij}\| \lesssim 1/n$.

The approximate linearization property describes that the deviation of the estimator $\widetilde{\mathbf{X}}$ after an appropriate orthogonal alignment $\mathbf{W}$ from the true value $\mathbf{X}_0$ can be approximately controlled by a linear combination of the centered Bernoulli random variables $(A_{ij} - \rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})_{i<j}$. Having defined the approximate linearization property that is required for the initial estimator $\widetilde{\mathbf{X}}$ of the one-step procedure (4.7), we present the following two theorems, which are the main technical results of this chapter.

Theorem 16 *Let* $\mathbf{A} \sim \mathrm{RDPG}(\mathbf{X}_0)$ *for some* $\mathbf{X}_0 = [\mathbf{x}_{01}, \ldots, \mathbf{x}_{0n}]^{\mathrm{T}} \in \mathcal{X}^n$ *with a sparsity factor* ρ_n*. Assume that condition* (4.1) *holds, and there exists some constant* $\delta > 0$ *such that* $\delta \le \min_{i,j} \mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j} \le \max_{i,j} \mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j} \le 1 - \delta$*. Denote* $\widehat{\mathbf{X}} = [\widehat{\mathbf{x}}_1, \ldots, \widehat{\mathbf{x}}_n]^{\mathrm{T}}$ *the one-step estimator defined by* (4.7) *based on an initial estimator* $\widetilde{\mathbf{X}} = [\widetilde{\mathbf{x}}_1, \ldots, \widetilde{\mathbf{x}}_n]^{\mathrm{T}}$ *satisfying the approximate linearization property* (4.8)*. Denote*

$$\mathbf{G}_n(\mathbf{x}) = \frac{1}{n}\sum_{j=1}^n \frac{\mathbf{x}_{0j}\mathbf{x}_{0j}^{\mathrm{T}}}{\mathbf{x}^{\mathrm{T}}\mathbf{x}_{0j}(1 - \rho_n\mathbf{x}^{\mathrm{T}}\mathbf{x}_{0j})}$$

for any $\mathbf{x} \in \mathcal{X}$ *such that* $\mathbf{x}^{\mathrm{T}}\mathbf{x}_{0j} \in [\delta, 1-\delta]$ *for all* $j \in [n]$*. If either* $\rho_n \equiv 1$ *for all* n *or* $\rho_n \to 0$ *but* $(\log n)^{2(1\vee\omega)}/(n\rho_n^5) \to 0$ *as* $n \to \infty$*, then there exists some orthogonal matrix* $\mathbf{W} \in \mathbb{O}(d)$ *such that*

$$\mathbf{W}^{\mathrm{T}}\widehat{\mathbf{x}}_i - \rho_n^{1/2}\mathbf{x}_{0i} = \frac{1}{n\sqrt{\rho_n}}\sum_{j=1}^n \frac{(A_{ij} - \rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})}{\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j}(1 - \rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})}\mathbf{G}_n(\mathbf{x}_{0i})^{-1}\mathbf{x}_{0j} + \widehat{\mathbf{R}}_i, \quad i = 1, \ldots, n, \tag{4.9}$$

where $\|\widehat{\mathbf{R}}_i\| = O_{\mathbb{P}_0}(n^{-1}\rho_n^{-5/2}(\log n)^{(1\vee\omega)})$ *and* $\sum_{i=1}^n \|\widehat{\mathbf{R}}_i\|^2 = O_{\mathbb{P}_0}((n\rho_n^5)^{-1}(\log n)^{2(1\vee\omega)})$*.*

Theorem 17 *Let* $\mathbf{A} \sim \mathrm{RDPG}(\mathbf{X}_0)$ *for some* $\mathbf{X}_0 = [\mathbf{x}_{01}, \ldots, \mathbf{x}_{0n}]^{\mathrm{T}} \in \mathcal{X}^n$ *with a sparsity factor* ρ_n*. Assume that the conditions of Theorem* 16 *hold, and denote* $\rho = \lim_{n\to\infty}\rho_n$*. Then there exists some orthogonal matrix* $\mathbf{W} \in \mathbb{O}(d)$ *such that as* $n \to \infty$*,*

$$\left\|\widehat{\mathbf{X}}\mathbf{W} - \rho_n^{1/2}\mathbf{X}_0\right\|_{\mathrm{F}}^2 \overset{\mathbb{P}_0}{\to} \int_{\mathcal{X}} \mathrm{tr}\left\{\mathbf{G}(\mathbf{x})^{-1}\right\} F(\mathrm{d}\mathbf{x}), \tag{4.10}$$

where

$$\mathbf{G}(\mathbf{x}) = \int_{\mathcal{X}} \frac{\mathbf{x}_1\mathbf{x}_1^{\mathrm{T}}}{\mathbf{x}^{\mathrm{T}}\mathbf{x}_1(1 - \rho\mathbf{x}^{\mathrm{T}}\mathbf{x}_1)} F(\mathrm{d}\mathbf{x}_1),$$

and for each fixed $i \in [n]$*,*

$$\sqrt{n}(\mathbf{W}^{\mathrm{T}}\widehat{\mathbf{x}}_i - \rho_n^{1/2}\mathbf{x}_{0i}) \overset{\mathcal{L}}{\to} \mathrm{N}(\mathbf{0}, \mathbf{G}(\mathbf{x}_{0i})^{-1}). \tag{4.11}$$

Since we have already shown that $\mathbf{\Sigma}(\mathbf{x}_{0i}) \succeq \mathbf{G}(\mathbf{x}_{0i})^{-1}$ for all $i \in [n]$, it follows that

$$\|\widehat{\mathbf{X}}\mathbf{W} - \rho_n^{1/2}\mathbf{X}_0\|_{\mathrm{F}}^2 - \|\widehat{\mathbf{X}}^{(\mathrm{ASE})}\mathbf{W} - \rho_n^{1/2}\mathbf{X}_0\|_{\mathrm{F}}^2$$

$$\overset{\mathbb{P}_0}{\to} \int_{\mathcal{X}} \mathrm{tr}\{\mathbf{\Sigma}(\mathbf{x}) - \mathbf{G}(\mathbf{x})^{-1}\}F(\mathrm{d}\mathbf{x}) = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n \mathrm{tr}\{\mathbf{\Sigma}(\mathbf{x}_{0i}) - \mathbf{G}(\mathbf{x}_{0i})^{-1}\} \geq 0,$$

and hence we conclude that the one-step estimator $\widehat{\mathbf{X}}$ improves the ASE $\widehat{\mathbf{X}}^{(\mathrm{ASE})}$ globally for all vertices asymptotically. Furthermore, locally for every fixed vertex $i \in [n]$, the ith row of the one-step estimator $\widehat{\mathbf{x}}_i$ is asymptotically efficient, in the sense that it has the same asymptotic covariance matrix with that of the maximum likelihood estimator as if the rest of the latent positions are known, and this covariance matrix is no greater than that of the corresponding row of the ASE in spectra.

Remark 9 *Theorem 16 asserts that the one-step estimator $\widehat{\mathbf{X}}$ dominates the ASE locally for each individual vertex and globally for all vertices, under the density assumption that $(n\rho_n^5)^{-1}(\log n)^{2(1\vee\omega)} \to 0$ as $n \to \infty$. When the graph is dense,* i.e., *$\rho_n \equiv 1$ for all n, it is easy to show that this condition holds. When ρ_n^{-1} is a polynomial of $\log n$, indicating that the graph is moderately sparse, this condition still holds. This condition starts to fail when the graph becomes very sparse,* e.g., *$\rho_n^{-1} \asymp n^t$ for some $t \geq 1/5$, in which case statistical inference becomes challenging due to the weak signal.*

It has been shown in [36] and [56] that for the ASE $\widehat{\mathbf{X}}^{(\mathrm{ASE})}$, there exists an orthogonal $\mathbf{W} \in \mathbb{O}(d)$ such that

$$\mathbf{W}^{\mathrm{T}}\widehat{\mathbf{x}}_i^{(\mathrm{ASE})} - \rho_n^{1/2}\mathbf{x}_{0i} = \rho_n^{-1/2}\sum_{j=1}^n (A_{ij} - \rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})[\mathbf{X}_0(\mathbf{X}_0^{\mathrm{T}}\mathbf{X}_0)^{-1}]_{j\cdot} + \widehat{\mathbf{R}}_i^{(\mathrm{ASE})},$$

where $[\mathbf{X}_0(\mathbf{X}_0^{\mathrm{T}}\mathbf{X}_0)^{-1}]_{j\cdot}$ denotes the vector formed by transposing the jth row of $\mathbf{X}_0(\mathbf{X}_0^{\mathrm{T}}\mathbf{X}_0)^{-1}$, and $(\sum_{i=1}^n \|\widehat{\mathbf{R}}_i^{(\mathrm{ASE})}\|_{\mathrm{F}}^2)^{1/2} = O_{\mathbb{P}_0}((n\rho_n)^{-1})$. Thus, the ASE satisfies the approximate linearization property (4.8) with $\omega = 0$ and $\boldsymbol{\zeta}_{ij} = [\mathbf{X}_0(\mathbf{X}_0^{\mathrm{T}}\mathbf{X}_0)^{-1}]_{j\cdot}$, and hence, $\widehat{\mathbf{X}}^{(\mathrm{ASE})}$ can be chosen as an initial estimator for the one-step procedure in practice.

Remark 10 *Theorem 17 implies that the one-step estimator $\widehat{\mathbf{X}}$ initialized with an estimator that satisfies the approximate linearization property (4.8) also satisfies (4.8) when the graph is dense ($\rho_n \equiv 1$ for all n). In this case, one can apply the one-step* *procedure multiple times, and the resulting estimator still has the same asymptotic behavior as given by Theorem 16. This multi-step updating strategy is of practical interest for more accurate estimation when the sample size is insufficient for asymptotic approximation.*

Proofs sketch for Theorem 16 and Theorem 17. The key for proving Theorem 16 and Theorem 17 is the formula (4.9). From here, we can apply the logarithmic Sobolev concentration inequality to (4.9) (see, for example, Section 6.4 in [69]) to show that $\|\widehat{\mathbf{X}}\mathbf{W} - \rho_n^{1/2}\mathbf{X}_0\|_{\mathrm{F}}^2$ converges in probability to its expectation, which is exactly (4.10). The asymptotic normality (4.11) of $\widehat{\mathbf{x}}_i$ can be obtained by directly applying the Lyapunov's central limit theorem to

$$\frac{1}{\sqrt{n\rho_n}}\sum_{j=1}^n \frac{(A_{ij} - \rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})}{\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j}(1 - \rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})}\mathbf{G}_n(\mathbf{x}_{0i})^{-1}\mathbf{x}_{0j},$$

which is a sum of independent random variables. We now sketch the derivation for (4.9). By the construction of the one-step estimator (4.7), we have,

$$\begin{aligned}\mathbf{W}^{\mathrm{T}}\widehat{\mathbf{x}}_i - \rho_n^{1/2}\mathbf{x}_{0i} &= \frac{1}{n\sqrt{\rho_n}}\sum_{j=1}^n \frac{(A_{ij} - \rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})}{\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j}(1 - \rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})}\mathbf{G}_n(\mathbf{x}_{0i})^{-1}\mathbf{x}_{0j} + (\mathbf{W}^{\mathrm{T}}\widetilde{\mathbf{x}}_i - \rho_n^{1/2}\mathbf{x}_{0i})\\ &\quad + \mathbf{G}_n(\mathbf{x}_{0i})^{-1}\mathbf{R}_{i1} + \mathbf{R}_{i2}\mathbf{R}_{i1} + \mathbf{R}_{i3},\end{aligned}$$

where

$$\begin{aligned}\mathbf{R}_{i1} &= \frac{1}{n\sqrt{\rho_n}}\sum_{j=1}^n\left\{\boldsymbol{\phi}_{ij}(\rho_n^{-1/2}\mathbf{W}^{\mathrm{T}}\widetilde{\mathbf{x}}_i, \rho_n^{-1/2}\mathbf{W}^{\mathrm{T}}\widetilde{\mathbf{x}}_j) - \boldsymbol{\phi}_{ij}(\mathbf{x}_{0i}, \mathbf{x}_{0j})\right\},\\ \boldsymbol{\phi}_{ij}(\mathbf{u},\mathbf{v}) &= \frac{(A_{ij} - \rho_n\mathbf{u}^{\mathrm{T}}\mathbf{v})\mathbf{v}}{\mathbf{u}^{\mathrm{T}}\mathbf{v}(1 - \rho_n\mathbf{u}^{\mathrm{T}}\mathbf{v})},\\ \mathbf{R}_{i2} &= \mathbf{W}^{\mathrm{T}}\left\{\frac{1}{n}\sum_{j=1}^n\frac{\widetilde{\mathbf{x}}_j\widetilde{\mathbf{x}}_j^{\mathrm{T}}}{\widetilde{\mathbf{x}}_i^{\mathrm{T}}\widetilde{\mathbf{x}}_j(1 - \widetilde{\mathbf{x}}_i^{\mathrm{T}}\widetilde{\mathbf{x}}_j)}\right\}^{-1}\mathbf{W} - \mathbf{G}_n(\mathbf{x}_{0i})^{-1},\\ \mathbf{R}_{i3} &= \frac{1}{n\sqrt{\rho_n}}\sum_{j=1}^n\frac{(A_{ij} - \rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})}{\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j}(1 - \rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})}\mathbf{R}_{i2}\mathbf{x}_{0j}.\end{aligned}$$

For $\mathbf{R}_{i1}$, we apply Taylor's expansion to $\boldsymbol{\phi}_{ij}$ together with the result

$$\|\widetilde{\mathbf{X}}\mathbf{W} - \rho_n^{1/2}\mathbf{X}_0\|_{2\to\infty} = O_{\mathbb{P}_0}\left(\frac{(\log n)^{(1\vee\omega)/2}}{\sqrt{n\rho_n}}\right),$$

which is a variation of Lemma 2.1 in [34], to obtain

$$\begin{aligned}
\mathbf{R}_{i1} &= \frac{1}{n\sqrt{\rho_n}}\sum_{j=1}^{n}\mathbb{E}_0\left\{\frac{\partial \boldsymbol{\phi}_{ij}}{\partial \mathbf{u}^{\mathrm{T}}}(\mathbf{x}_{0i},\mathbf{x}_{0j})\right\}(\rho_n^{-1/2}\mathbf{W}^{\mathrm{T}}\widetilde{\mathbf{x}}_i - \mathbf{x}_{0i}) + O_{\mathbb{P}_0}\left(\frac{(\log n)^{1\vee\omega}}{n\rho_n^{5/2}}\right)\\
&= -\mathbf{G}_n(\mathbf{x}_{0i})(\mathbf{W}^{\mathrm{T}}\widetilde{\mathbf{x}}_i - \rho_n^{1/2}\mathbf{x}_{0i}) + O_{\mathbb{P}_0}\left(\frac{(\log n)^{1\vee\omega}}{n\rho_n^{5/2}}\right).
\end{aligned}$$

For $\mathbf{R}_{i2}$, we directly obtain from $\|\widetilde{\mathbf{X}}\mathbf{W} - \rho_n^{1/2}\mathbf{X}_0\|_{2\to\infty} = O_{\mathbb{P}_0}((n\rho_n)^{-1/2}(\log n)^{(1\vee\omega)/2})$ and the Lipschitz continuity of the function $(\mathbf{u},\mathbf{v}) \mapsto \{\mathbf{u}^{\mathrm{T}}\mathbf{v}(1-\rho_n\mathbf{u}^{\mathrm{T}}\mathbf{v})\}^{-1}\mathbf{v}\mathbf{v}^{\mathrm{T}}$ to conclude that $\|\mathbf{R}_{i2}\| = O_{\mathbb{P}_0}(\rho_n^{-1}n^{-1/2}(\log n)^{(1\vee\omega)/2})$. Finally, an application of Hoeffding's inequality in conjunction with the union bound yields $\|\mathbf{R}_{i3}\| = O_{\mathbb{P}_0}(\rho_n^{-3/2}n^{-1}(\log n)^{1\vee\omega})$. Thus we obtain that

$$\mathbf{W}^{\mathrm{T}}\widehat{\mathbf{x}}_i - \rho_n^{1/2}\mathbf{x}_{0i} = \frac{1}{n\sqrt{\rho_n}}\sum_{j=1}^{n}\frac{(A_{ij}-\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})}{\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j}(1-\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})}\mathbf{G}_n(\mathbf{x}_{0i})^{-1}\mathbf{x}_{0j} + O_{\mathbb{P}_0}\left(\frac{(\log n)^{1\vee\omega}}{n\rho_n^{5/2}}\right).$$

The result $\sum_{i=1}^{n}\|\widehat{\mathbf{R}}_i\|^2 = O_{\mathbb{P}_0}((n\rho_n^5)^{-1}(\log n)^{2(1\vee\omega)})$ follows a similar but more technical argument. The detailed proof is provided in Section 5.3.

4.4 Estimating the population Laplacian spectral embedding

In addition to the ASE that directly works on the observed adjacency matrix $\mathbf{A}$, another popular technique for statistical analysis on random graphs is based on the normalized Laplacian of $\mathbf{A}$, which is particularly useful for clustering in stochastic block models [37, 38]. Formally, the normalized Laplacian of a matrix $\mathbf{M}$ with non-negative entries, denoted by $\mathcal{L}(\mathbf{M})$, is defined by

$$\mathcal{L}(\mathbf{M}) = (\mathrm{diag}(\mathbf{M}\mathbf{1}))^{-1/2}\mathbf{M}(\mathrm{diag}(\mathbf{M}\mathbf{1}))^{-1/2},$$

where, given $\mathbf{z} = [z_1,\ldots,z_n]^{\mathrm{T}} \in \mathbb{R}^n$, $\mathrm{diag}(\mathbf{z})$ is the $n\times n$ diagonal matrix with $z_1,\ldots,z_n$ being its diagonal entries. Here we follow the definition of normalized Laplacian adopted in [56] in contrast to the combinatorial Laplacian $\mathrm{diag}(\mathbf{M}\mathbf{1}) - \mathbf{M}$ that has been applied to graph theory (see, for example, [70]). For the adjacency matrix $\mathbf{A}$, the (i,j) entry of the normalized Laplacian matrix $\mathcal{L}(\mathbf{A})$ can be viewed as the connection between vertices i and j normalized by the square-roots of the degrees of these two vertices.

Recall that the edge probability matrix $\rho_n\mathbf{X}\mathbf{X}^{\mathrm{T}}$ is a positive semidefinite low-rank matrix when $\mathbf{A}\sim\mathrm{RDPG}(\mathbf{X})$ with a sparsity factor ρ_n. Similarly, the normalized Laplacian of $\rho_n\mathbf{X}\mathbf{X}^{\mathrm{T}}$ can also be viewed as a positive semidefinite low-rank matrix:

$$\mathcal{L}(\rho_n\mathbf{X}\mathbf{X}^{\mathrm{T}}) = (\mathrm{diag}(\mathbf{X}\mathbf{X}^{\mathrm{T}}\mathbf{1}))^{-1/2}\mathbf{X}\mathbf{X}^{\mathrm{T}}(\mathrm{diag}(\mathbf{X}\mathbf{X}^{\mathrm{T}}\mathbf{1}))^{-1/2} = \mathbf{Y}\mathbf{Y}^{\mathrm{T}},$$

where $\mathbf{Y} = [\mathbf{y}_1,\dots,\mathbf{y}_n]^{\mathrm{T}}\in\mathbb{R}^{n\times d}$, and $\mathbf{y}_i = \mathbf{x}_i(\sum_{j=1}^n \mathbf{x}_i\mathbf{x}_j)^{-1/2}$. We see immediately that $\mathbf{Y}$ matrix is exactly a reparameterization of the latent position matrix $\mathbf{X}$, and is referred to as the *population Laplacian spectral embedding*. Following the same spirit of the formulation of the ASE through (4.2), one can analogously define the *sample Laplacian spectral embedding* (LSE) $\breve{\mathbf{X}}$ of $\mathbf{A}$ into $\mathbb{R}^d$ by solving the least-squares problem [37]

$$\breve{\mathbf{X}} = \underset{\mathbf{Y}\in\mathbb{R}^{n\times d}}{\arg\min}\,\|\mathcal{L}(\mathbf{A}) - \mathbf{Y}\mathbf{Y}^{\mathrm{T}}\|_{\mathrm{F}}^2. \tag{4.12}$$

We will make the convention that the LSE refers to the sample LSE, and the population LSE will be used to refer to the $\mathbf{Y}$ matrix specifically. Similar to the ASE $\widehat{\mathbf{X}}^{(\mathrm{ASE})}$, the consistency and asymptotic distribution results hold for $\breve{\mathbf{X}}$ as an estimator for $\mathbf{Y}$ as well:

Theorem 18 (Theorems 3.1 and 3.2 of [56]) *Let* $\mathbf{A}\sim\mathrm{RDPG}(\mathbf{X}_0)$ *with a sparsity factor* ρ_n *for some* $\mathbf{X}_0 = [\mathbf{x}_{01},\dots,\mathbf{x}_{0n}]^{\mathrm{T}}\in\mathcal{X}^n$. *Assume that condition* (4.1) *holds. Suppose either* $\rho_n\equiv 1$ *for all* n *or* $\rho_n\to 0$ *but* $(\log n)^4/(n\rho_n)\to 0$ *as* $n\to\infty$, *and denote* $\rho = \lim_{n\to\infty}\rho_n$. *Let* $\breve{\mathbf{X}} = [\breve{\mathbf{x}}_1,\dots,\breve{\mathbf{x}}_n]^{\mathrm{T}}$ *be the LSE of* $\mathbf{A}$ *into* $\mathbb{R}^d$ *defined by* (4.12). *Define the following quantities:*

$$\mathbf{Y}_0 = [\mathbf{y}_{01},\dots,\mathbf{y}_{0n}]^{\mathrm{T}},\quad \mathbf{y}_{0i} = \frac{\mathbf{x}_{0i}}{\sqrt{\sum_{j=1}^n \mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j}}},\quad \boldsymbol{\mu} = \int_{\mathcal{X}}\mathbf{x}F(\mathrm{d}\mathbf{x}),\quad \widetilde{\boldsymbol{\Delta}} = \int_{\mathcal{X}}\frac{\mathbf{x}\mathbf{x}^{\mathrm{T}}}{\mathbf{x}^{\mathrm{T}}\boldsymbol{\mu}}F(\mathrm{d}\mathbf{x}),$$

$$\widetilde{\mathbf{\Sigma}}(\mathbf{x}) = \frac{1}{\boldsymbol{\mu}^{\mathrm{T}}\mathbf{x}}\left(\widetilde{\mathbf{\Delta}}^{-1} - \frac{\mathbf{x}\boldsymbol{\mu}^{\mathrm{T}}}{2\boldsymbol{\mu}^{\mathrm{T}}\mathbf{x}}\right)\left[\int_{\mathcal{X}}\left\{\frac{\mathbf{x}^{\mathrm{T}}\mathbf{x}_1(1-\rho\mathbf{x}^{\mathrm{T}}\mathbf{x}_1)}{(\boldsymbol{\mu}^{\mathrm{T}}\mathbf{x}_1)^2}\mathbf{x}_1\mathbf{x}_1^{\mathrm{T}}\right\}F(\mathrm{d}\mathbf{x}_1)\right]\left(\widetilde{\mathbf{\Delta}}^{-1} - \frac{\mathbf{x}\boldsymbol{\mu}^{\mathrm{T}}}{2\boldsymbol{\mu}^{\mathrm{T}}\mathbf{x}}\right)^{\mathrm{T}}.$$

Then there exists an orthogonal $\mathbf{W} \in \mathbb{R}^{d\times d}$ such that as $n \to \infty$,

$$n\rho_n\|\breve{\mathbf{X}}\mathbf{W} - \mathbf{Y}_0\|_{\mathrm{F}}^2 \overset{a.s.}{\to} \int \mathrm{tr}\{\widetilde{\mathbf{\Sigma}}(\mathbf{x})\}F(\mathrm{d}\mathbf{x}), \tag{4.13}$$

and for any fixed $i \in [n]$,

$$n\rho_n^{1/2}(\mathbf{W}^{\mathrm{T}}\breve{\mathbf{x}}_i - \mathbf{y}_{0i}) \overset{\mathcal{L}}{\to} \mathrm{N}(\mathbf{0}, \widetilde{\mathbf{\Sigma}}(\mathbf{x}_{0i})). \tag{4.14}$$

Similar to the ASE, the LSE is also a least-squares type estimator and does not involve the likelihood function. Therefore, to utilize the likelihood information of the sampling model for estimating the population LSE, we propose the following one-step estimator $\widehat{\mathbf{Y}}$ for $\mathbf{Y}_0$ based on the one-step estimator $\widehat{\mathbf{X}} = [\widehat{\mathbf{x}}_1, \ldots, \widehat{\mathbf{x}}_n]^{\mathrm{T}}$ defined in (4.7) and an initial estimator $\widetilde{\mathbf{X}} = [\widetilde{\mathbf{x}}_1, \ldots, \widetilde{\mathbf{x}}_n]^{\mathrm{T}}$ that satisfies the approximate linearization property (4.8):

$$\widehat{\mathbf{Y}} = [\widehat{\mathbf{y}}_1, \ldots, \widehat{\mathbf{y}}_n]^{\mathrm{T}}, \quad \widehat{\mathbf{y}}_i = \frac{\widehat{\mathbf{x}}_i}{\sqrt{\sum_{j=1}^n \widehat{\mathbf{x}}_i^{\mathrm{T}}\widetilde{\mathbf{x}}_j}}, \quad i = 1, 2, \ldots, n. \tag{4.15}$$

The likelihood information is thus absorbed into $\widehat{\mathbf{Y}}$ through the one-step estimator $\widehat{\mathbf{X}}$ for $\mathbf{X}_0$. We characterize the global and local behavior of the one-step estimator $\widehat{\mathbf{Y}}$ for the normalized Laplacian matrix via the following two theorems.

Theorem 19 *Let $\mathbf{A} \sim \mathrm{RDPG}(\mathbf{X}_0)$ for some $\mathbf{X}_0 = [\mathbf{x}_{01}, \ldots, \mathbf{x}_{0n}]^{\mathrm{T}} \in \mathcal{X}^n$ with a sparsity factor ρ_n. Assume that the condition (4.1) holds and the conditions of Theorem 16 hold. Denote $\widehat{\mathbf{Y}} = [\widehat{\mathbf{y}}_1, \ldots, \widehat{\mathbf{y}}_n]^{\mathrm{T}}$ the one-step estimator for the normalized Laplacian matrix defined by (4.15), and $\boldsymbol{\mu}_n = (1/n)\sum_{i=1}^n \mathbf{x}_{0i}$. If $(\log n)^{2(1\vee\omega)}(n\rho_n^6)^{-1} \to 0$, then there exists some orthogonal matrix $\mathbf{W} \in \mathbb{O}(d)$ such that*

$$\sqrt{n}(\mathbf{W}^{\mathrm{T}}\widehat{\mathbf{y}}_i - \mathbf{y}_{0i}) = \rho_n^{-1/2}\frac{1}{\sqrt{\boldsymbol{\mu}_n^{\mathrm{T}}\mathbf{x}_{0i}}}\left(\mathbf{I}_d - \frac{\mathbf{x}_{0i}\boldsymbol{\mu}_n^{\mathrm{T}}}{2\boldsymbol{\mu}_n^{\mathrm{T}}\mathbf{x}_{0i}}\right)(\mathbf{W}^{\mathrm{T}}\widehat{\mathbf{x}}_i - \rho_n^{1/2}\mathbf{x}_{0i}) + \mathbf{R}_i^{(\mathrm{L})}$$

for $i = 1, 2, \ldots, n$, where $\|\mathbf{R}_i^{(\mathrm{L})}\| = O_{\mathbb{P}_0}((n\rho_n^3)^{-1}(\log n)^{1\vee\omega})$ and $\sum_{i=1}^n \|\mathbf{R}_i^{(\mathrm{L})}\|^2 = O_{\mathbb{P}_0}\left((n\rho_n^6)^{-1}(\log n)^{2(1\vee\omega)}\right)$.

Theorem 20 *Let* $\mathbf{A} \sim \mathrm{RDPG}(\mathbf{X}_0)$ *for some* $\mathbf{X}_0 = [\mathbf{x}_{01}, \ldots, \mathbf{x}_{0n}]^{\mathrm{T}} \in \mathcal{X}^n$ *with a sparsity factor* ρ_n. *Assume the conditions of Theorem 19 hold. Denote* $\widehat{\mathbf{Y}} = [\widehat{\mathbf{y}}_1, \ldots, \widehat{\mathbf{y}}_n]^{\mathrm{T}}$ *the one-step estimator for the normalized Laplacian matrix defined by (4.15), and*

$$\widetilde{\mathbf{G}}(\mathbf{x}) = \frac{1}{(\boldsymbol{\mu}^{\mathrm{T}}\mathbf{x})}\left(\mathbf{I}_d - \frac{\mathbf{x}\boldsymbol{\mu}^{\mathrm{T}}}{2\boldsymbol{\mu}^{\mathrm{T}}\mathbf{x}}\right)\mathbf{G}(\mathbf{x})^{-1}\left(\mathbf{I}_d - \frac{\mathbf{x}\boldsymbol{\mu}^{\mathrm{T}}}{2\boldsymbol{\mu}^{\mathrm{T}}\mathbf{x}}\right)^{\mathrm{T}}$$

for any $\mathbf{x} \in \mathcal{X}$ *such that* $\mathbf{x}^{\mathrm{T}}\mathbf{x}_{0j} \in [\delta, 1-\delta]$ *for any* $j \in [n]$, *where* $\boldsymbol{\mu} = \int_{\mathcal{X}} \mathbf{x} F(\mathrm{d}\mathbf{x})$, *and* $\mathbf{G}(\cdot)$ *is defined in Theorem 17. Then there exists some orthogonal matrix* $\mathbf{W} \in \mathbb{O}(d)$ *such that*

$$n\rho_n \left\|\widehat{\mathbf{Y}}\mathbf{W} - \mathbf{Y}_0\right\|_{\mathrm{F}}^2 \xrightarrow{\mathbb{P}_0} \int_{\mathcal{X}} \mathrm{tr}\left\{\widetilde{\mathbf{G}}(\mathbf{x})\right\} F(\mathrm{d}\mathbf{x}), \tag{4.16}$$

and for each fixed $i \in [n]$,

$$n\rho_n^{1/2}(\mathbf{W}^{\mathrm{T}}\widehat{\mathbf{y}}_i - \mathbf{y}_{0i}) \xrightarrow{\mathcal{L}} \mathrm{N}(\mathbf{0}, \widetilde{\mathbf{G}}(\mathbf{x}_{0i})). \tag{4.17}$$

In Section 4.3, it is shown that the one-step estimator $\widehat{\mathbf{X}}$ dominates the ASE $\widehat{\mathbf{X}}^{(\mathrm{ASE})}$ for estimating $\mathbf{X}_0$ asymptotically. Here we argue that $\widehat{\mathbf{Y}}$ dominates the LSE $\breve{\mathbf{X}}$ asymptotically as well. Denote $\boldsymbol{\Lambda} = \mathrm{diag}\{(\boldsymbol{\mu}_n^{\mathrm{T}}\mathbf{x}_{01})^{-1}, \ldots, (\boldsymbol{\mu}_n^{\mathrm{T}}\mathbf{x}_{0n})^{-1}\}$ and

$$\widetilde{\boldsymbol{\Delta}}_n = \int_{\mathcal{X}} \left(\frac{\mathbf{x}_1\mathbf{x}_1^{\mathrm{T}}}{\boldsymbol{\mu}_n^{\mathrm{T}}\mathbf{x}_1}\right) F_n(\mathrm{d}\mathbf{x}_1) = \frac{1}{n}\sum_{j=1}^n \frac{\mathbf{x}_{0j}\mathbf{x}_{0j}^{\mathrm{T}}}{\boldsymbol{\mu}_n^{\mathrm{T}}\mathbf{x}_{0j}} = \frac{1}{n}\mathbf{X}_0^{\mathrm{T}}\boldsymbol{\Lambda}\mathbf{X}_0 \to \widetilde{\boldsymbol{\Delta}}.$$

Suppose $\mathbf{X}_0$ yields the singular value decomposition $\mathbf{X}_0 = \mathbf{U}_0\mathbf{S}_0^{1/2}\mathbf{V}_0^{\mathrm{T}}$ with $\mathbf{U}_0 \in \mathbb{O}(n,d)$, $\mathbf{S}_0^{1/2}$ being diagonal, and $\mathbf{V}_0 \in \mathbb{O}(d)$. By Corollary 2.1 in [71] we have

$$(\mathbf{U}_0^{\mathrm{T}}\boldsymbol{\Lambda}\mathbf{U}_0)(\mathbf{U}_0^{\mathrm{T}}\mathbf{D}(\mathbf{x})\mathbf{U}_0)(\mathbf{U}_0^{\mathrm{T}}\boldsymbol{\Lambda}\mathbf{U}_0) \succeq \mathbf{U}_0^{\mathrm{T}}\boldsymbol{\Lambda}\mathbf{D}(\mathbf{x})\boldsymbol{\Lambda}\mathbf{U}_0,$$

implying that

$$\begin{aligned}
&\widetilde{\boldsymbol{\Delta}}_n^{-1}\left(\frac{1}{n}\mathbf{X}_0^{\mathrm{T}}\boldsymbol{\Lambda}\mathbf{D}(\mathbf{x})\boldsymbol{\Lambda}\mathbf{X}_0\right)\widetilde{\boldsymbol{\Delta}}_n^{-1}\\
&\quad = n(\mathbf{X}_0^{\mathrm{T}}\boldsymbol{\Lambda}\mathbf{X}_0)^{-1}\left(\mathbf{X}_0^{\mathrm{T}}\boldsymbol{\Lambda}\mathbf{D}(\mathbf{x})\boldsymbol{\Lambda}\mathbf{X}_0\right)(\mathbf{X}_0^{\mathrm{T}}\boldsymbol{\Lambda}\mathbf{X}_0)^{-1}\\
&\quad = n\mathbf{V}_0\mathbf{S}_0^{-1/2}(\mathbf{U}_0^{\mathrm{T}}\boldsymbol{\Lambda}\mathbf{U}_0)^{-1}(\mathbf{U}_0^{\mathrm{T}}\boldsymbol{\Lambda}\mathbf{D}(\mathbf{x})\boldsymbol{\Lambda}\mathbf{U}_0)(\mathbf{U}_0^{\mathrm{T}}\boldsymbol{\Lambda}\mathbf{U}_0)^{-1}\mathbf{S}_0^{-1/2}\mathbf{V}_0^{\mathrm{T}}\\
&\quad \succeq n\mathbf{V}_0\mathbf{S}_0^{-1/2}(\mathbf{U}_0^{\mathrm{T}}\mathbf{D}(\mathbf{x})^{-1}\mathbf{U}_0)^{-1}\mathbf{S}_0^{-1/2}\mathbf{V}_0^{\mathrm{T}}
\end{aligned}$$

$$
\begin{aligned}
&= n(\mathbf{V}_0\mathbf{S}_0^{1/2}\mathbf{U}_0^{\mathrm{T}}\mathbf{D}(\mathbf{x})^{-1}\mathbf{U}_0\mathbf{S}_0^{1/2}\mathbf{V}_0^{\mathrm{T}})^{-1}\\
&= \left(\frac{1}{n}\mathbf{X}_0^{\mathrm{T}}\mathbf{D}(\mathbf{x})^{-1}\mathbf{X}_0\right)^{-1}.
\end{aligned}
$$

Since $\widetilde{\boldsymbol{\Delta}}_n\boldsymbol{\mu}_n = \boldsymbol{\mu}_n$, it follows that

$$
\begin{aligned}
\widetilde{\boldsymbol{\Sigma}}_n(\mathbf{x}) &:= \frac{1}{\boldsymbol{\mu}^{\mathrm{T}}\mathbf{x}}\left(\widetilde{\boldsymbol{\Delta}}_n^{-1} - \frac{\mathbf{x}\boldsymbol{\mu}_n^{\mathrm{T}}}{2\boldsymbol{\mu}_n^{\mathrm{T}}\mathbf{x}}\right)\left\{\frac{1}{n}\sum_{j=1}^n\frac{\mathbf{x}^{\mathrm{T}}\mathbf{x}_{0j}(1-\rho_n\mathbf{x}^{\mathrm{T}}\mathbf{x}_{0j})}{(\boldsymbol{\mu}^{\mathrm{T}}\mathbf{x}_{0j})^2}\mathbf{x}_{0j}\mathbf{x}_{0j}^{\mathrm{T}}\right\}\left(\widetilde{\boldsymbol{\Delta}}_n^{-1} - \frac{\mathbf{x}\boldsymbol{\mu}_n^{\mathrm{T}}}{2\boldsymbol{\mu}_n^{\mathrm{T}}\mathbf{x}}\right)^{\mathrm{T}}\\
&= \frac{1}{\boldsymbol{\mu}^{\mathrm{T}}\mathbf{x}}\left(\widetilde{\boldsymbol{\Delta}}_n^{-1} - \frac{\mathbf{x}\boldsymbol{\mu}_n^{\mathrm{T}}\widetilde{\boldsymbol{\Delta}}_n^{-1}}{2\boldsymbol{\mu}_n^{\mathrm{T}}\mathbf{x}}\right)\left(\frac{1}{n}\mathbf{X}_0^{\mathrm{T}}\boldsymbol{\Lambda}\mathbf{D}(\mathbf{x})\boldsymbol{\Lambda}\mathbf{X}_0\right)\left(\widetilde{\boldsymbol{\Delta}}_n^{-1} - \frac{\mathbf{x}\boldsymbol{\mu}_n^{\mathrm{T}}\widetilde{\boldsymbol{\Delta}}_n^{-1}}{2\boldsymbol{\mu}_n^{\mathrm{T}}\mathbf{x}}\right)^{\mathrm{T}}\\
&= \frac{1}{\boldsymbol{\mu}^{\mathrm{T}}\mathbf{x}}\left(\mathbf{I}_d - \frac{\mathbf{x}\boldsymbol{\mu}_n^{\mathrm{T}}}{2\boldsymbol{\mu}_n^{\mathrm{T}}\mathbf{x}}\right)\widetilde{\boldsymbol{\Delta}}_n^{-1}\left(\frac{1}{n}\mathbf{X}_0^{\mathrm{T}}\boldsymbol{\Lambda}\mathbf{D}(\mathbf{x})\boldsymbol{\Lambda}\mathbf{X}_0\right)\widetilde{\boldsymbol{\Delta}}_n^{-1}\left(\mathbf{I}_d - \frac{\mathbf{x}\boldsymbol{\mu}_n^{\mathrm{T}}}{2\boldsymbol{\mu}_n^{\mathrm{T}}\mathbf{x}}\right)^{\mathrm{T}}\\
&\succeq \frac{1}{\boldsymbol{\mu}^{\mathrm{T}}\mathbf{x}}\left(\mathbf{I}_d - \frac{\mathbf{x}\boldsymbol{\mu}_n^{\mathrm{T}}}{2\boldsymbol{\mu}_n^{\mathrm{T}}\mathbf{x}}\right)\left(\frac{1}{n}\mathbf{X}_0^{\mathrm{T}}\mathbf{D}(\mathbf{x})^{-1}\mathbf{X}_0\right)^{-1}\left(\mathbf{I}_d - \frac{\mathbf{x}\boldsymbol{\mu}_n^{\mathrm{T}}}{2\boldsymbol{\mu}_n^{\mathrm{T}}\mathbf{x}}\right)^{\mathrm{T}} \to \widetilde{\mathbf{G}}(\mathbf{x})
\end{aligned}
$$

as $n\to\infty$. Therefore, $\widetilde{\boldsymbol{\Sigma}}(\mathbf{x}) = \lim_{n\to\infty}\boldsymbol{\Sigma}_n(\mathbf{x}) \succeq \widetilde{\mathbf{G}}(\mathbf{x})$. This shows that locally for vertex i, the one-step estimator $\widehat{\mathbf{Y}}$ improves the LSE $\check{\mathbf{X}}$ asymptotically in terms of smaller asymptotic covariance matrix in spectra. In addition,

$$
n\rho_n\|\widehat{\mathbf{Y}}\mathbf{W} - \mathbf{Y}_0\|_{\mathrm{F}}^2 - n\rho_n\|\check{\mathbf{X}}\mathbf{W} - \mathbf{Y}_0\|_{\mathrm{F}}^2 \overset{\mathbb{P}_0}{\to} \int_{\mathcal{X}}\mathrm{tr}\{\widetilde{\boldsymbol{\Sigma}}(\mathbf{x}) - \widetilde{\mathbf{G}}(\mathbf{x})\}F(\mathrm{d}\mathbf{x}) \geq 0.
$$

Namely, the one-step estimator $\widehat{\mathbf{Y}}$ also improves the LSE globally for all vertices in terms of the sum of squared errors $\|\widehat{\mathbf{Y}}\mathbf{W} - \mathbf{Y}_0\|_{\mathrm{F}}^2$.

4.5 Numerical examples

4.5.1 A two-block stochastic block model example

We first consider the following rank-one stochastic block model with two communities on n vertices. The block probability matrix is given by

$$
\mathbf{B} = \begin{bmatrix} p^2 & pq \\ pq & q^2 \end{bmatrix},
$$

where $p, q \in (0,1)$, and the cluster assignment function $\tau : [n] \to [2]$ satisfies

$$
\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n \mathbb{1}\{\tau(i) = 1\} = \pi_1, \quad \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^n\{\tau(i) = 2\} = \pi_2, \quad \text{where } \pi_1 + \pi_2 = 1.
$$

The distribution F satisfying condition (4.1) can be explicitly computed: $F(\mathrm{d}x) = \pi_1\delta_p(\mathrm{d}x) + \pi_2\delta_q(\mathrm{d}x)$ with $\pi_1 + \pi_2 = 1$, $p, q \in (0, 1)$. We focus on clustering the vertices as a subsequent inference task after obtaining the estimates for the latent positions, or the population LSE. To estimate the latent positions or their transformations, we compute the following four estimates: the ASE (4.2), the one-step estimator (4.7) initialized at the ASE, abbreviated as OSE-A, the LSE (4.12), and the one-step estimator (4.15) for the population LSE, abbreviated as OSE-L. These estimated latent positions, or their transformations corresponding to the population LSE, are then used as input features for clustering vertices.

Our goal is to compare the performance of clustering using these four estimates. Therefore, a criterion that is independent of the choice of the clustering algorithm, but focuses on the distributions of the input features, is needed. To this end, we introduce the concept of *minimum pairwise Chernoff distance*. Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ be i.i.d. following a distribution $F \in \{F_1, \ldots, F_K\}$, where $F_k(\mathrm{d}\mathbf{x}) = f_k(\mathbf{x})\mathrm{d}\mathbf{x}$, $k \in [K]$, and suppose the task is to determine whether $F = F_k$ for $k \in [K]$. Assume that $F = F_k$ with prior probability π_k, $k \in [K]$. Then for any decision rule u, the risk of u is $r(u) = \sum_{k=1}^{K} \pi_k \sum_{l \neq k} p_{kl}(u)$, where $p_{kl}(u)$ is the probability that the decision rule u assigns $F = F_l$ when the true underlying distribution is $F = F_k$. In the context of vertex clustering, the decision rule u plays the role of a clustering algorithm, and $\mathbf{x}_i$'s are treated as either one of the aforementioned four estimates. Since we are interested in a criterion that does not depend on u, it is natural to investigate the behavior of the risk when the optimal decision rule is applied. The following result characterized the optimal error rate [72]:

$$\inf_u \lim_{n \to \infty} \frac{1}{n} r(u) = -\min_{k \neq l} C(F_k, F_l),$$

where $C(F_k, F_l)$ is the *Chernoff information* between F_k and F_l defined by the authors of [73, 74]

$$C(F_k, F_l) = \sup_{t\in(0,1)} \left\{ -\log \int f_k(\mathbf{x})^t f_l(\mathbf{x})^{1-t}(\mathrm{d}\mathbf{x}) \right\}, \tag{4.18}$$

and $\min_{k\neq l} C(F_k, F_l)$ is the minimum pairwise Chernoff distance. This quantity describes the asymptotic decaying rate of the error for the optimal decision rule, with larger values indicating a lower optimal error rate. In our context, since the asymptotic distributions of the four estimators are multivariate normal, it is useful to derive the Chernoff information for two multivariate normal distributions:

$$C(F_k, F_l) = \sup_{t\in(0,1)} \left\{ \frac{t(1-t)}{2}(\boldsymbol{\mu}_k - \boldsymbol{\mu}_l)^{\mathrm{T}}\mathbf{V}_t^{-1}(\boldsymbol{\mu}_k - \boldsymbol{\mu}_l) + \frac{1}{2}\log\frac{|\mathbf{V}_t|}{|\mathbf{V}_k|^t|\mathbf{V}_l|^{1-t}} \right\},$$

where $F_k = \mathrm{N}(\boldsymbol{\mu}_k, \mathbf{V}_k)$ and $F_l = \mathrm{N}(\boldsymbol{\mu}_l, \mathbf{V}_l)$, and $\mathbf{V}_t = t\mathbf{V}_k + (1-t)\mathbf{V}_l$. Note that the term $(1/2)\log\{|\mathbf{V}_t|/(|\mathbf{V}_k|^t|\mathbf{V}_l|^{1-t})\}$ is negligible for sufficiently large n.

For a K-block stochastic block model with a positive semidefinite block probability matrix $\mathbf{B} = (\mathbf{X}_0^*)(\mathbf{X}_0^*)^{\mathrm{T}}$, where $\mathbf{X}_0^* = [\mathbf{x}_{01}^*, \ldots, \mathbf{x}_{0K}^*]^{\mathrm{T}} \in \mathbb{R}^{K\times d}$, $d \leq K$, and a cluster assignment function $\tau : [n] \to [K]$ satisfying $(1/n)\sum_{i=1}^n \mathbb{1}\{\tau(i) = k\} \to \pi_k$ for $k \in [K]$ and $\sum_{k=1}^K \pi_k = 1$, we define the following quantities for the ASE, the LSE, the OSE-A, and the OSE-L, respectively:

$$\begin{aligned}
\rho^*_{\mathrm{ASE}} &= \min_{k\neq l}\sup_{t\in(0,1)} \frac{nt(1-t)}{2}(\mathbf{x}_{0k}^* - \mathbf{x}_{0l}^*)^{\mathrm{T}}\boldsymbol{\Sigma}_{kl}^{-1}(t)(\mathbf{x}_{0k}^* - \mathbf{x}_{0l}^*),\\
\rho^*_{\mathrm{LSE}} &= \min_{k\neq l}\sup_{t\in(0,1)} \frac{n^2t(1-t)}{2}(\mathbf{y}_{0k}^* - \mathbf{y}_{0l}^*)^{\mathrm{T}}\widetilde{\boldsymbol{\Sigma}}_{kl}^{-1}(t)(\mathbf{y}_{0k}^* - \mathbf{y}_{0l}^*),\\
\rho^*_{\mathrm{OSE-A}} &= \min_{k\neq l}\sup_{t\in(0,1)} \frac{nt(1-t)}{2}(\mathbf{x}_{0k}^* - \mathbf{x}_{0l}^*)^{\mathrm{T}}\mathbf{G}_{kl}^{-1}(t)(\mathbf{x}_{0k}^* - \mathbf{x}_{0l}^*),\\
\rho^*_{\mathrm{OSE-L}} &= \min_{k\neq l}\sup_{t\in(0,1)} \frac{n^2t(1-t)}{2}(\mathbf{y}_{0k}^* - \mathbf{y}_{0l}^*)^{\mathrm{T}}\widetilde{\mathbf{G}}_{kl}^{-1}(t)(\mathbf{y}_{0k}^* - \mathbf{y}_{0l}^*),
\end{aligned}$$

where

$$\begin{aligned}
&\boldsymbol{\Sigma}_{kl}(t) = t\boldsymbol{\Sigma}(\mathbf{x}_{0k}^*) + (1-t)\boldsymbol{\Sigma}(\mathbf{x}_{0l}^*), &&\widetilde{\boldsymbol{\Sigma}}_{kl}(t) = t\widetilde{\boldsymbol{\Sigma}}(\mathbf{x}_{0k}^*) + (1-t)\widetilde{\boldsymbol{\Sigma}}(\mathbf{x}_{0l}^*),\\
&\mathbf{G}_{kl}(t) = t\mathbf{G}(\mathbf{x}_{0k}^*)^{-1} + (1-t)\mathbf{G}(\mathbf{x}_{0l}^*)^{-1}, &&\widetilde{\mathbf{G}}_{kl}(t) = t\widetilde{\mathbf{G}}_k(\mathbf{x}_{0k}^*) + (1-t)\widetilde{\mathbf{G}}(\mathbf{x}_{0l}^*),
\end{aligned}$$

and $\mathbf{y}_{0k}^* = \mathbf{x}_{0k}^*\{\sum_{l=1}^K n\pi_k(\mathbf{x}_{0k}^*)^{\mathrm{T}}(\mathbf{x}_{0l}^*)\}^{-1/2}$. These quantities are motivated by the use of the minimum pairwise Chernoff distance for measuring clustering performance. Note that for all $t \in (0,1)$, we have seen in Section 4.3 and Section 4.4 that

$$\mathbf{\Sigma}_{kl}(t) = t\mathbf{\Sigma}(\mathbf{x}_{0k}^*) + (1-t)\mathbf{\Sigma}(\mathbf{x}_{0l}^*) \succeq t\mathbf{G}(\mathbf{x}_{0k}^*)^{-1} + (1-t)\mathbf{G}(\mathbf{x}_{0l}^*)^{-1} = \mathbf{G}_{kl}(t),$$
$$\widetilde{\mathbf{\Sigma}}_{kl}(t) = t\widetilde{\mathbf{\Sigma}}(\mathbf{x}_{0k}^*) + (1-t)\widetilde{\mathbf{\Sigma}}(\mathbf{x}_{0l}^*) \succeq t\widetilde{\mathbf{G}}(\mathbf{x}_{0k}^*) + (1-t)\widetilde{\mathbf{G}}(\mathbf{x}_{0l}^*) = \widetilde{\mathbf{G}}_{kl}(t).$$

It follows that $\rho^*_{\mathrm{ASE}} \leq \rho^*_{\mathrm{OSE-A}}$ and $\rho^*_{\mathrm{LSE}} \leq \rho^*_{\mathrm{OSE-L}}$ regardless of the choice of the underlying true latent positions. Namely, the decaying rate of the optimal decision error using the OSE-A is always smaller than that using the ASE, and the same holds for the comparison between the OSE-L and the LSE. We also note that the above criteria are independent of the choice of the clustering algorithm and only depend on the distribution of the input features.

Specialized to the two-block stochastic block model example considered in this subsection, we obtain by simple algebra that

$$\rho^*_{\mathrm{OSE-A}} = \frac{n(p-q)^2}{2}\{G(p)^{-1/2} + G(q)^{-1/2}\}^{-2},$$
$$\rho^*_{\mathrm{OSE-L}} = \frac{n(p-q)^2}{2}\left\{\frac{\sqrt{p}+\sqrt{q}}{2\sqrt{p}}G(p)^{-1/2} + \frac{\sqrt{p}+\sqrt{q}}{2\sqrt{q}}G(q)^{-1/2}\right\}^{-2}.$$

where

$$G(p) = \frac{\pi_1 p^2}{p^2(1-p^2)} + \frac{\pi_2 q^2}{pq(1-pq)}, \quad G(q) = \frac{\pi_1 p^2}{pq(1-pq)} + \frac{\pi_2 q^2}{q^2(1-q^2)}.$$

In particular, when $p \neq q$, $\rho^*_{\mathrm{OSE-A}} < \rho^*_{\mathrm{OSE-L}}$ if and only if $q > p$. Namely, when $q < p$, the OSE-A dominates the OSE-L in terms of the optimal error rate, and when $q > p$, the OSE-L outperforms the OSE-A. To visualize this result, we fix $\pi_1 = 0.6, \pi_2 = 0.4$, let p range over $[0.2, 0.8]$, $r = q - p$ range over in $[-0.15, 0.15]\backslash\{0\}$, compute the ratio $\rho^*_{\mathrm{OSE-A}}/\rho^*_{\mathrm{OSE-L}}$, and plot the numerical results in Figure 4-1.

Besides the aforementioned large sample conclusion, we perform two finite-sample experiments. We first compute the four estimates with $n = 200$, $p = 0.6, q = 0.4$ and $n = 200$, $p = 0.45$, $q = 0.6$, respectively. In each of the scenarios, we choose the Gaussian-mixture-model-based (GMM-based) clustering algorithm [75, 76], which is

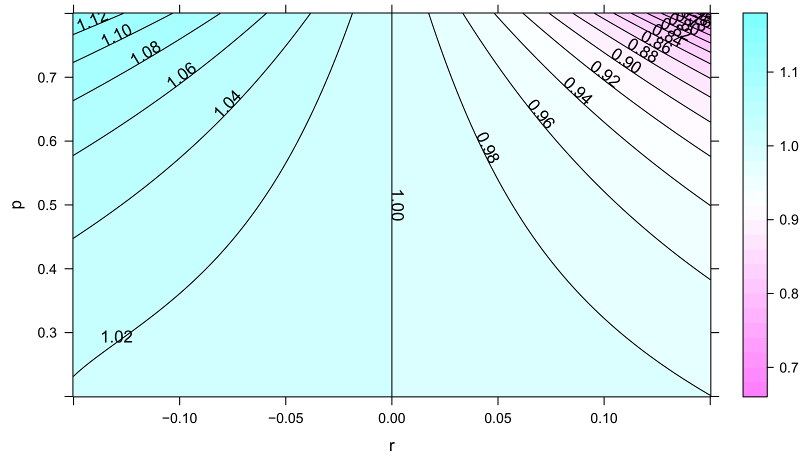


Figure 4-1. Heatmap and level curves of the ratio $\rho^*_{\text{OSE-A}}/\rho^*_{\text{OSE-L}}$ for $p \in [0.2, 0.8]$ and $r \in [-0.15, 0.15]\backslash\{0\}$ for the two-block stochastic block model example.

recommended in [56], for subsequent vertex clustering task using these estimates as features. To evaluate the finite-sample experimental clustering results, we follow the same idea in Section 3.7 and adopt the Rand index to measure the agreement between any two partitions of the vertices $[n]$. Table 4-I reports the Rand indices to evaluate the accuracy of the clustering results in comparison with the underlying true cluster assignment, which are averaged over 100 Monte Carlo replicates. The results are in accordance with the aforementioned large sample conclusion.

Table 4-I. The two-block stochastic block model example: Rand indices of the GMM-based clustering algorithm using different estimates. For each setup $p = 0.6, q = 0.4$ and $p = 0.45$, $q = 0.6$, the Rand indices are averaged over 100 Monte Carlo replicates of adjacency matrices.

Estimates	ASE	OSE-A	LSE	OSE-L
$p = 0.6$, $q = 0.4$	0.9049	**0.9083**	0.9023	0.9038
$p = 0.45$, $q = 0.6$	0.7771	0.7790	0.7840	**0.7853**

Remark 11 *Unlike the minimum pairwise Chernoff distance, which is an asymptotic criterion for comparing the performance of different estimators in terms of the subsequent optimal clustering task and does not depend on the clustering algorithm, the Rand index can only reflect the behavior of the clustering result and may depend on*

the clustering method we choose.

4.5.2 A three-block stochastic block model example

Next, we consider the following three-block stochastic block model on n vertices with the block probability matrix $\mathbf{B} = (\mathbf{X}_0^*)(\mathbf{X}_0^*)^{\mathrm{T}}$, where

$$(\mathbf{X}_0^*)^{\mathrm{T}} = \begin{bmatrix} 0.3 & 0.3 & 0.6 \\ 0.3 & 0.6 & 0.3 \end{bmatrix},$$

and a cluster assignment function $\tau : [n] \to [3]$, such that as $n \to \infty$,

$$\frac{1}{n}\sum_{i=1}^{n} \mathbb{1}\{\tau(i) = 1\} \to 0.3, \quad \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}\{\tau(i) = 2\} \to 0.3, \quad \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}\{\tau(i) = 3\} \to 0.4.$$

The corresponding distribution F satisfying condition (4.1) is $F(\mathrm{d}\mathbf{x}) = \sum_{k=1}^{3} \pi_k \delta_{\mathbf{x}_{0k}^*}(\mathrm{d}\mathbf{x})$, where $\pi_1 = \pi_2 = 0.3$, $\pi_3 = 0.4$, $\mathbf{x}_{01}^* = [0.3, 0.3]^{\mathrm{T}}$, $\mathbf{x}_{02}^* = [0.3, 0.6]^{\mathrm{T}}$, and $\mathbf{x}_{03}^* = [0.6, 0.3]^{\mathrm{T}}$. For each $n \in \{500, 600, \ldots, 1200\}$, we generate 100 replicates of the simulated adjacency matrices from the above sampling model and then estimate the latent positions or the population LSE by the following four methods: the ASE (4.2), the one-step estimator (4.7) initialized at the ASE (OSE-A), the LSE (4.12), and the one-step estimator (4.15) for the population LSE (OSE-L). The goal is to compare the performance of vertex clustering with the GMM-based clustering algorithm applied to these estimates.

Table 4-II lists the Rand indices of the GMM-based clustering applied to the four estimates in comparison with the underlying true cluster assignment, and these Rand indices are averaged over the 100 Monte Carlo replicates. When the number of vertices $n \in \{500, 600, \ldots, 900\}$, the clustering results based on the ASE outperform the rest competitors. However, as n increases with $n \geq 900$, the best result is given by either the OSE-A or the OSE-L. In particular, when $n \in \{1100, 1200\}$, the OSE-A and the OSE-L yield better results than the ASE and the LSE, respectively. These numerical results are in accordance with the fact that the ASE and the LSE are dominated by the OSE-A and OSE-L asymptotically, respectively.

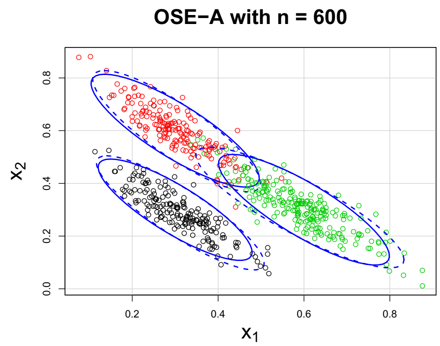
Table 4-II. The three-block stochastic block model example: Rand indices of the GMM-based clustering algorithm using different estimates. The number of vertices n ranges over $\{500, 600, \ldots, 1200\}$, and for each n, the Rand indices are averaged over 100 Monte Carlo replicates of adjacency matrices.

Estimates	ASE	OSE-A	LSE	OSE-L
$n = 500$	**0.89753**	0.88890	0.82177	0.89184
$n = 600$	**0.93518**	0.93200	0.89716	0.93164
$n = 700$	**0.95635**	0.95494	0.93754	0.95519
$n = 800$	**0.96880**	0.96860	0.95777	0.96863
$n = 900$	0.97628	**0.97651**	0.96975	0.97641
$n = 1000$	0.98381	0.98359	0.97850	**0.98384**
$n = 1100$	0.98918	**0.98940**	0.98600	0.98936
$n = 1200$	0.99166	**0.99173**	0.98920	0.99160

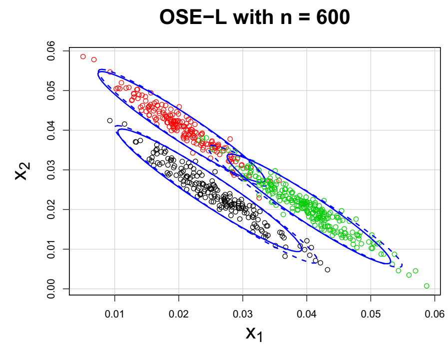
For each $n \in \{600, 900, 1200\}$, we also compute the OSE-A $\widehat{\mathbf{X}}$ and the OSE-L $\widehat{\mathbf{Y}}$ for each block, as well as the corresponding cluster-specific sample covariance matrices after applying the appropriate orthogonal transformation towards the underlying true $\mathbf{X}_0$ and $\mathbf{Y}_0$, for one randomly selected instance among the 100 replicated adjacency matrices. The results are tabulated in Table 4-III and Table 4-IV, respectively, in comparison with the limit covariance matrices given by Theorem 17 and Theorem 20. It can be seen that as n increases, the sample covariance matrices converge to their corresponding cluster-specific limit covariance matrices. The scatter points of $\widehat{\mathbf{X}}$ and $\widehat{\mathbf{Y}}$ after applying the orthogonal alignment matrix $\mathbf{W}$ towards $\mathbf{X}_0$ and $\mathbf{Y}_0$ are visualized in Figure 4-2, along with the cluster-specific 95% empirical and asymptotic confidence ellipses in dashed lines and solid lines, respectively. These figures also validate the aforementioned limit results.

4.5.3 Wikipedia graph data

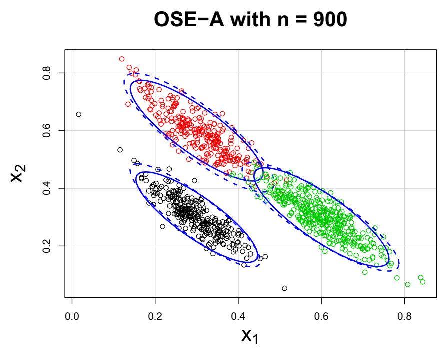
We finally investigate the performance of the proposed one-step estimation procedure to a real-world Wikipedia graph dataset, which is available at http://www.cis.jhu.edu/~parky/Data/data.html. The Wikipedia graph dataset consists of an adjacency matrix among $n = 1382$ Wikipedia articles that are within two hyperlinks of the article



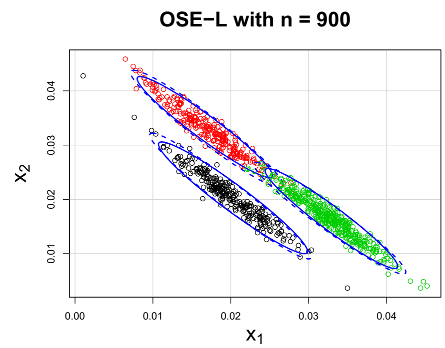
OSE-A with $n = 600$



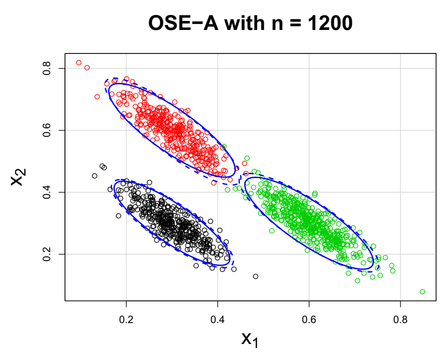
OSE-L with $n = 600$



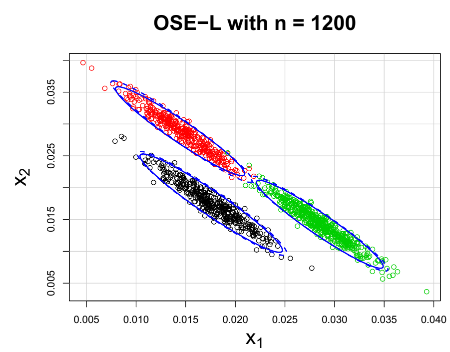
OSE-A with $n = 900$



OSE-L with $n = 900$



OSE-A with $n = 1200$



OSE-L with $n = 1200$

Figure 4-2. Scatter plots of the OSE-A and OSE-L in the three-block stochastic block model example with n vertices, with $n \in \{600, 900, 1200\}$. The scatter points are colored according to the cluster assignment of the corresponding vertices. For each specific cluster, the 95% empirical confidence ellipses are displayed by the dashed lines, along with the 95% asymptotic confidence ellipses drawn using the solid lines, as provided by Theorem 17 and Theorem 20.

Table 4-III. Three-block stochastic block model example: the cluster-specific sample covariance matrices for the OSE-A with the number of vertices $n \in \{600, 900, 1200\}$.

k	$n = 600$	$n = 900$	$n = 1200$	Limit covariance
1	$\begin{bmatrix} 3.76 & -3.65 \\ -3.65 & 4.62 \end{bmatrix}$	$\begin{bmatrix} 3.65 & -3.67 \\ -3.67 & 4.76 \end{bmatrix}$	$\begin{bmatrix} 3.55 & -3.16 \\ -3.16 & 4.03 \end{bmatrix}$	$\begin{bmatrix} 3.22 & -2.90 \\ -2.90 & 3.70 \end{bmatrix}$
2	$\begin{bmatrix} 3.76 & -3.80 \\ -3.80 & 5.21 \end{bmatrix}$	$\begin{bmatrix} 4.85 & -4.70 \\ -4.70 & 6.30 \end{bmatrix}$	$\begin{bmatrix} 4.40 & -4.43 \\ -4.43 & 5.92 \end{bmatrix}$	$\begin{bmatrix} 3.84 & -3.52 \\ -3.52 & 4.59 \end{bmatrix}$
3	$\begin{bmatrix} 5.81 & -4.75 \\ -4.75 & 5.18 \end{bmatrix}$	$\begin{bmatrix} 5.32 & -4.58 \\ -4.58 & 5.30 \end{bmatrix}$	$\begin{bmatrix} 4.66 & -4.14 \\ -4.14 & 5.05 \end{bmatrix}$	$\begin{bmatrix} 3.96 & -3.50 \\ -3.50 & 4.41 \end{bmatrix}$

Table 4-IV. Three-block stochastic block model example: the cluster-specific sample covariance matrices for the OSE-L with the number of vertices $n \in \{600, 900, 1200\}$.

k	$n = 600$	$n = 900$	$n = 1200$	Limit covariance
1	$\begin{bmatrix} 14.90 & -15.75 \\ -15.75 & 17.79 \end{bmatrix}$	$\begin{bmatrix} 14.41 & -15.61 \\ -15.61 & 18.02 \end{bmatrix}$	$\begin{bmatrix} 13.55 & -14.03 \\ -14.03 & 15.88 \end{bmatrix}$	$\begin{bmatrix} 12.41 & -12.78 \\ -12.78 & 14.37 \end{bmatrix}$
2	$\begin{bmatrix} 10.30 & -11.05 \\ -11.05 & 12.82 \end{bmatrix}$	$\begin{bmatrix} 12.96 & -13.73 \\ -13.73 & 15.79 \end{bmatrix}$	$\begin{bmatrix} 12.12 & -13.09 \\ -13.09 & 15.15 \end{bmatrix}$	$\begin{bmatrix} 10.23 & -10.48 \\ -10.48 & 11.74 \end{bmatrix}$
3	$\begin{bmatrix} 13.65 & -13.32 \\ -13.32 & 14.05 \end{bmatrix}$	$\begin{bmatrix} 12.79 & -12.98 \\ -12.98 & 14.21 \end{bmatrix}$	$\begin{bmatrix} 11.37 & -11.81 \\ -11.81 & 13.30 \end{bmatrix}$	$\begin{bmatrix} 9.82 & -10.16 \\ -10.16 & 11.51 \end{bmatrix}$

"Algebraic Geometry", and these articles are further manually labeled according to one of the following 6 descriptions: People, Places, Dates, Things, Math, and Category. To determine a suitable embedding dimension d for the random dot product graph model, we follow the ad-hoc approach of [77] and computes

$$\widehat{d} = \underset{d=1,2,\ldots,q}{\arg\max} \left\{ \sum_{k=1}^{d} \log f(\sigma_k(\mathbf{A}); \widehat{\mu}_1, \widehat{\sigma}^2) + \sum_{k=d+1}^{q} \log f(\sigma_k(\mathbf{A}); \widehat{\mu}_2, \widehat{\sigma}^2) \right\},$$

where $f(x; \mu, \sigma^2) = (2\pi\sigma^2)^{-1/2} \exp\{-(x-\mu)^2/(2\sigma^2)\}$ is the normal density with mean μ and variance σ^2,

$$\mu_1 = \frac{1}{d}\sum_{k=1}^{d} \sigma_k(\mathbf{A}), \quad \mu_2 = \frac{1}{p-d}\sum_{k=d+1}^{p} \sigma_k(\mathbf{A}), \quad \widehat{\sigma}^2 = \frac{(d-1)s_1^2 + (p-d-1)s_2^2}{p-2},$$

s_1^2, s_2^2 are the sample variances of $\{\sigma_k(\mathbf{A})\}_{k=1}^{d}$ and $\{\sigma_k(\mathbf{A})\}_{k=d+1}^{q}$, respectively, and q is an upper bound for the embedding dimension. Here we select $q = 50$ as a conservative upper bound, resulting in $\widehat{d} = 11$.

We then compute the ASE, the LSE, the OSE-A, and the OSE-L, with the embedding dimension $d = 11$, and then apply the GMM-based clustering algorithm to these estimates with the number of clusters being 6. We compare the similarity between the manually assigned 6 class labels and these clustering results by computing the respective Rand indices, which are tabulated in Table 4-V. The results show that the one-step procedure for the population LSE outperforms the rest competitors, as it provides the clustering result that is most similar to the original class label assignment among the four methods.

Table 4-V. Wikipedia Graph Data: Rand indices of the GMM-based clustering algorithm applied to the ASE, the LSE, the OSE-A, and the OSE-L, respectively, with the number of clusters being 6, in comparison with the corresponding manual labels.

Method	ASE	LSE	OSE-A	OSE-L
Rand Index	0.7429	0.7350	0.7413	**0.7538**

Table 4-VI. Wikipedia Graph Data: Rand indices of the GMM-based clustering algorithm applied to the ASE, the LSE, the OSE-A, and the OSE-L, respectively, with the number of clusters being 2, in comparison with the corresponding one-versus-all manual labels for the class "Dates".

Method	ASE	LSE	OSE-A	OSE-L
Rand Index	0.5289	0.5097	**0.5432**	0.5313

Besides evaluating the performance of the overall clustering for the 6 manually-assigned labels, we also specifically focus on the comparison of the article class "Dates" against the rest articles. We apply the GMM-based clustering algorithm to the aforementioned four estimates again, but with the number of clusters being 2, and report the Rand indices in Table 4-VI. We can see that the proposed one-step procedure improves the clustering accuracy as well when we focus on the comparison between the article class "Dates" against the rest labels. The scatter plots of the first-versus-second dimension of the four estimates are visualized in Figure 4-3, along with the cluster-specific 95% empirical confidence ellipses in dashed lines. It can be

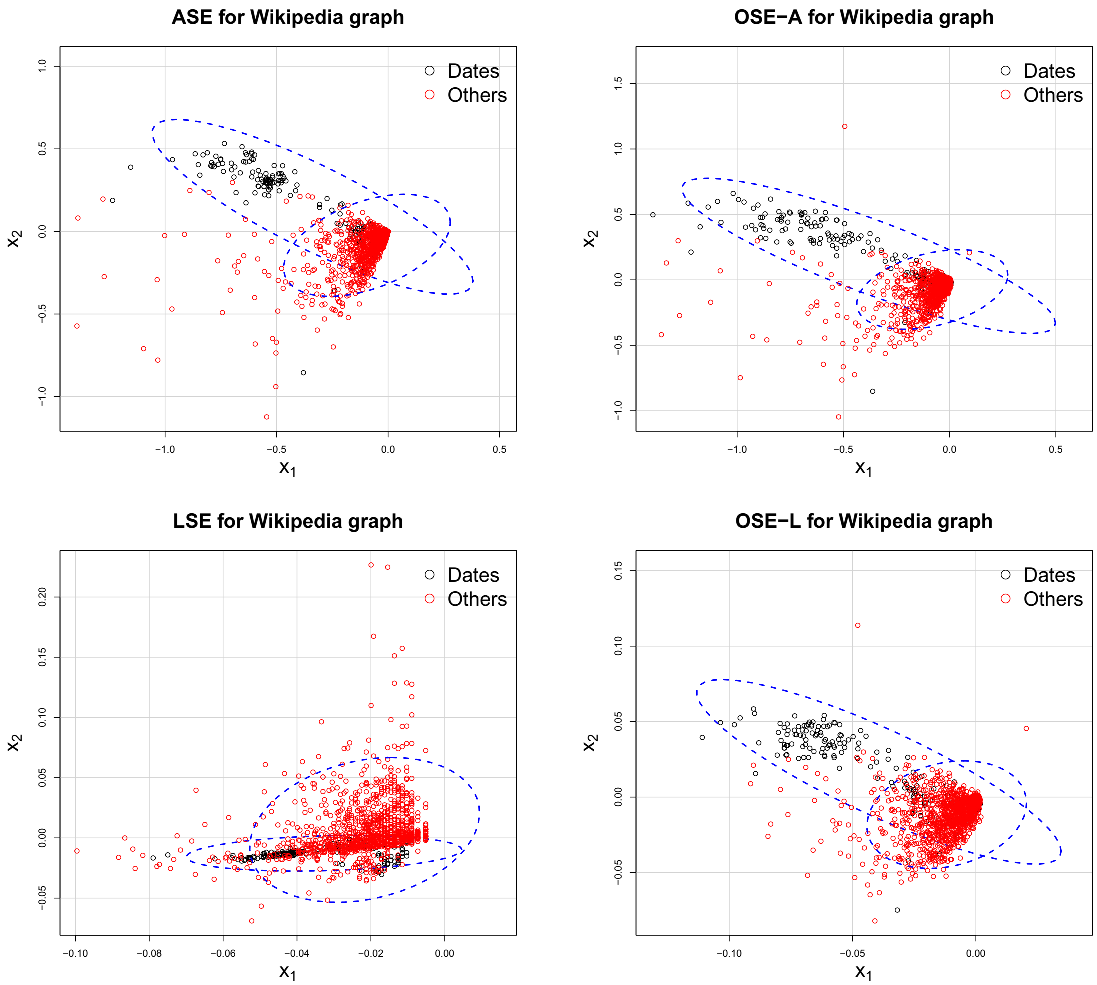


Figure 4-3. Wikipedia graph data: The scatter plots of the first-versus-second dimension of the four estimates. The scatter points are colored according to whether the articles are in the class "Dates" or the others. The 95% empirical cluster-specific confidence ellipses are displayed by the dashed lines.

clearly seen that the OSE-A and the OSE-L outperform the ASE and the LSE, as the one-step procedure results in better separation of the articles in the "Dates" class from the rest articles.

4.6 Discussion

We assume that the embedding dimension d for the random dot product graph is known throughout the chapter. The proposed one-step procedure is also valid when the true dimension d for the underlying sampling model is unknown, but the method proceeds by first finding the ASE into $\mathbb{R}^{d'}$ for some $d' \geq 1$ and $d' < d$ (*i.e.*, when the dimension is under-estimated), and then computing the one-step estimator based on d'. Our Theorem 17 and Theorem 20 still hold and can be easily proved, as suggested by the authors of [56]. On the other hand, leveraging Bayesian methods when the dimension d is unknown is a promising future direction in light of the recent progress in Bayesian theory and methods for low-rank matrix models with undetermined rank [48, 55] and network models [78, 2].

We have shown that the one-step procedure produces an estimator enjoying fascinating asymptotic properties both globally for all vertices and locally for each individual vertex. Nevertheless, for problems with relatively small sample sizes, we found in simulation examples that the one-step estimators do not necessarily provide us with better numerical results compared to the classical adjacency/Laplacian spectral embedding. Since the one-step procedure is exactly a single iteration of the Newton-Raphson algorithm with the observed Hessian matrix replaced by the negative Fisher information matrix, we hope to develop an iterative algorithm for finding a local maximum of the likelihood function by repeating the one-step procedure multiple times until convergence. Such an iterative algorithm can be implemented in conjunction with the regularization of the Fisher information matrix and backtracking procedure for finding suitable step sizes to achieve faster convergence [79]. Furthermore, developing a scalable version of such an algorithm will be highly desirable for the emerging big-data and large graphs. It will also be useful to explore the statistical properties of the estimator obtained by the iterative algorithm and establish its theoretical guarantee. We defer these research topics to future work.

Chapter 5

Proofs and auxiliary technical results

5.1 Proofs for Chapter 2

5.1.1 Proofs of results in Section 2.4.1

Proof of Lemma 2. Recall that $\pi(b_{jk} \mid \xi_j = 1) = (\lambda/2)\mathrm{e}^{-\lambda|b_{jk}|}$ follows the Laplace distribution with scale parameter $1/\lambda$, and that the Laplace distribution can be alternatively represented as a normal-variance mixture distribution as follows:

$$(b_{jk} \mid \xi_j = 1, \phi_{jk}) \sim \mathrm{N}\left(0, \frac{\phi_{jk}}{\lambda^2}\right), \quad \text{and} \quad \phi_{jk} \sim \mathrm{Exp}(1/2).$$

On the other hand, by the prior construction $(|b_{jk}| \mid \xi_j = 0, \lambda_0) \sim \mathrm{Gamma}(1/r, \lambda_0 + \lambda)$, it follows that $(\|\mathbf{B}_{j*}\|_1 \mid \xi_j = 0, \lambda_0) \sim \mathrm{Exp}(\lambda_0 + \lambda)$. Denote $S_0 = \mathrm{supp}(\mathbf{B}_0)$. Now we construct the following event

$$\mathcal{B} = \bigcap_{j \in S_0} \{\xi_j = 1, 1 \leq \phi_{jk} \leq 2, k \in [r]\} \cap \bigcap_{j \in S_0^c} \{\xi_j = 0\} \cap \left\{\lambda_0 + \lambda \geq \frac{\sqrt{2p}}{\eta}\left(\log \frac{p}{s}\right)\right\}$$

and denote $\boldsymbol{\phi} = [\phi_{jk} : j \in S_0, k \in [r]]_{s \times r}$.

Step 1: Conditioning on the event $\mathcal{B}$. For any $(\boldsymbol{\phi}, \boldsymbol{\xi}, \lambda_0) \in \mathcal{B}$, we use a union bound to derive

$$\Pi\left(\|\mathbf{B} - \mathbf{B}_0\|_{\mathrm{F}} < \eta \,\middle|\, \boldsymbol{\phi}, \boldsymbol{\xi}, \lambda_0\right)$$

$$
\begin{aligned}
&\geq \Pi\left(\sum_{j\in S_0}\|\mathbf{B}_{j*}-\mathbf{B}_{0j*}\|_2^2<\frac{\eta^2}{2}\;\middle|\;\boldsymbol{\phi},\boldsymbol{\xi},\lambda_0\right)\prod_{j\in S_0^c}\Pi\left(\|\mathbf{B}_{j*}\|_1\leq\frac{\eta}{\sqrt{2p}}\;\middle|\;\boldsymbol{\phi},\boldsymbol{\xi},\lambda_0\right)\\
&\geq \Pi\left(\sum_{j\in S_0}\|\mathbf{B}_{j*}-\mathbf{B}_{0j*}\|_2^2<\frac{\eta^2}{2}\;\middle|\;\boldsymbol{\phi},\boldsymbol{\xi},\lambda_0\right)\prod_{j\in S_0^c}\left[1-\exp\left\{-\frac{(\lambda_0+\lambda)\eta}{\sqrt{2p}}\right\}\right]\\
&\geq \Pi\left(\sum_{j\in S_0}\|\mathbf{B}_{j*}-\mathbf{B}_{0j*}\|_2^2<\frac{\eta^2}{2}\;\middle|\;\boldsymbol{\phi},\boldsymbol{\xi},\lambda_0\right)\left\{\left(1-\frac{s}{p}\right)^{p/s}\right\}^s\\
&\geq \Pi\left(\sum_{j\in S_0}\|\mathbf{B}_{j*}-\mathbf{B}_{0j*}\|_2^2<\frac{\eta^2}{2}\;\middle|\;\boldsymbol{\phi},\boldsymbol{\xi},\lambda_0\right)\exp\{-\log(2\mathrm{e})s\},
\end{aligned}
$$

where the last inequality is due to the fact that $(1-x)^{1/x}\geq\exp\{-\log(2\mathrm{e})\}$ when $x\in[0,1/2]$. It then suffices to provide a lower bound for the first factor. We take advantage of the fact that $(b_{jk}\mid\xi_j=1,\phi_{jk})\sim\mathrm{N}(0,\phi_{jk}/\lambda^2)$ and apply Anderson's lemma (see, for example, Lemma 1.4 in the supporting document of [41]) together with the union bound to derive

$$
\begin{aligned}
&\Pi\left(\sum_{j\in S_0}\|\mathbf{B}_{j*}-\mathbf{B}_{0j*}\|_2^2<\frac{\eta^2}{2}\;\middle|\;\boldsymbol{\phi},\boldsymbol{\xi},\lambda_0\right)\\
&\quad\geq\exp\left(-\frac{1}{2}\sum_{j\in S_0}\sum_{k=1}^r\frac{\lambda^2b_{0jk}^2}{\phi_{jk}}\right)\Pi\left(\sum_{j\in S_0}\sum_{k=1}^r b_{jk}^2<\frac{\eta^2}{2}\;\middle|\;\boldsymbol{\phi},\boldsymbol{\xi},\lambda_0\right)\\
&\quad\geq\exp\left(-\frac{1}{2}\sum_{j\in S_0}\sum_{k=1}^r\lambda^2b_{0jk}^2\right)\prod_{j\in S_0}\prod_{k=1}^r\Pi\left(\frac{\lambda^2b_{jk}^2}{\phi_{jk}}<\frac{\lambda^2\eta^2}{2\phi_{jk}rs}\;\middle|\;\boldsymbol{\phi},\boldsymbol{\xi},\lambda_0\right)\\
&\quad\geq\exp\left(-\frac{\lambda^2}{2}\sum_{j\in S_0}\|\mathbf{B}_0\|_{2\to\infty}^2\right)\prod_{j\in S_0}\prod_{k=1}^r\left\{2\Phi\left(\frac{\lambda\eta}{2\sqrt{rs}}\right)-1\right\}\\
&\quad\geq\exp\left(-\frac{\lambda^2}{2}s\|\mathbf{B}_0\|_{2\to\infty}^2-sr-sr\left|\log\frac{\lambda\eta}{2\sqrt{rs}}\right|\right),
\end{aligned}
$$

where the fact that $\log\{2\Phi(x)-1\}\geq-1-\log(x)$ for small $x>0$ is applied in the last inequality.

Step 2: Control the prior probability of the event $\mathcal{B}$. Recall that

$$
\mathcal{B}=\bigcap_{j\in S_0}\{\xi_j=1,1\leq\phi_{jk}\leq2,k\in[r]\}\cap\bigcap_{j\in S_0^c}\{\xi_j=0\}\cap\left\{\lambda_0+\lambda\geq\frac{\sqrt{2p}}{\eta}\left(\log\frac{p}{s}\right)\right\}.
$$

Then conditioning on θ, we obtain by construction

$$
\Pi(\mathcal{B})=\left\{\prod_{j\in S_0}\prod_{k=1}^r\Pi(1\leq\phi_{jk}\leq2)\right\}\left\{\int_0^1\theta^s(1-\theta)^{p-s}\Pi(\mathrm{d}\theta)\right\}\Pi\left\{\lambda_0+\lambda\geq\frac{\sqrt{2p}}{\eta}\left(\log\frac{p}{s}\right)\right\}
$$

$$\geq \exp(-3sr)\left\{\int_0^1 \theta^s(1-\theta)^{p-s}\Pi(\mathrm{d}\theta)\right\}\Pi\left\{\lambda_0 \geq \frac{\sqrt{2p}}{\eta}\left(\log\frac{p}{s}\right)\right\}.$$

We first focus on the third factor. By assumption $\eta > 1/p^\gamma$ for some $\gamma > 0$, implying that

$$\Pi\left\{\lambda_0 > \frac{\sqrt{2p}}{\eta}\left(\log\frac{p}{s}\right)\right\} \geq \Pi(\lambda_0 > p^\gamma) = 1 - \frac{1}{\Gamma(1/p^2)}\int_{1/p^\gamma}^\infty x^{1/p^2-1}\mathrm{e}^{-x}\mathrm{d}x.$$

Using an inequality for the incomplete Gamma function [80]:

$$\int_{4\delta}^\infty x^{-1}\mathrm{e}^{-x/2}\mathrm{d}x = \int_{2\delta}^\infty x^{-1}\mathrm{e}^{-x}\mathrm{d}x \leq \log\frac{1}{\delta}$$

for small values of $\delta > 0$, and the fact that $\Gamma(x) \geq 1$ when $0 < x \leq 1$, we have:

$$1 - \frac{1}{\Gamma(1/p^2)}\int_{1/p^\gamma}^\infty x^{1/p^2-1}\mathrm{e}^{-x}\mathrm{d}x \geq 1 - \int_{4/(4p^\gamma)}^\infty x^{-1}\mathrm{e}^{-x/2}\mathrm{d}x \geq 1 - \log\frac{1}{4p^\gamma} \geq \mathrm{e}^{-1}$$

for sufficiently large p (sufficiently small η). Next we consider the second factor. Write

$$\begin{aligned}
\int_0^1 \theta^s(1-\theta)^{p-s}\Pi(\mathrm{d}\theta) &\geq \int_{s/p^{1+\kappa}}^{2s/p^{1+\kappa}} \exp\left\{-s\log\left(\frac{1-\theta}{\theta}\right) - p\log\left(\frac{1}{1-\theta}\right)\right\}\Pi(\mathrm{d}\theta) \\
&\geq \int_{s/p^{1+\kappa}}^{2s/p^{1+\kappa}} \exp\left\{-s\log\left(\frac{p^{1+\kappa}}{s}\right) - p\log\left(1 + \frac{s}{p^{1+\kappa}-s}\right)\right\}\Pi(\mathrm{d}\theta) \\
&\geq \exp\left\{-(\kappa+1)s\log p - 2s\right\}\Pi\left(\frac{s}{p^{1+\kappa}} \leq \theta \leq \frac{2s}{p^{1+\kappa}}\right)
\end{aligned}$$

for sufficiently large p. Observe that

$$\begin{aligned}
\Pi\left(\frac{s}{p^{1+\kappa}} \leq \theta \leq \frac{2s}{p^{1+\kappa}}\right) &= \Pi\left(\frac{p^{1+\kappa}-2s}{p^{1+\kappa}} \leq 1-\theta \leq \frac{p^{1+\kappa}-s}{p^{1+\kappa}}\right) \\
&\geq \frac{1}{4p^{1+\kappa}}\left(1 - \frac{2s}{p^{1+\kappa}}\right)^{4p^{1+\kappa}-1}\left(\frac{s}{p^{1+\kappa}}\right) \\
&\geq \frac{1}{4p^{1+\kappa}}\left\{\left(1 - \frac{2s}{p^{1+\kappa}}\right)^{p^{1+\kappa}/(2s)}\right\}^{8s}\left(\frac{s}{p^{1+\kappa}}\right) \\
&\geq \exp\left\{-(2\kappa+19)s\log p\right\},
\end{aligned}$$

we conclude that $\Pi(\mathcal{B}) \geq \exp\left\{-3sr - 1 - (3\kappa+22)s\log p\right\}$.

Lower bound prior concentration by restricting over $\mathcal{B}$: We complete the proof by restricting over the event $\mathcal{B}$ as follows:

$$\Pi\left(\|\mathbf{B}-\mathbf{B}_0\|_{\mathrm{F}} < \eta\right) \geq \mathbb{E}_\Pi\left\{\Pi\left(\|\mathbf{B}-\mathbf{B}_0\|_{\mathrm{F}} < \eta \,\middle|\, \boldsymbol{\phi}, \boldsymbol{\xi}, \lambda_0\right)\mathbb{1}(\mathcal{B})\right\}$$

$$
\begin{aligned}
&\geq \left\{ \inf_{(\boldsymbol{\phi},\boldsymbol{\xi},\lambda_0)\in\mathcal{B}} \Pi\left(\|\mathbf{B}-\mathbf{B}_0\|_{\mathrm{F}} < \eta \,\middle|\, \boldsymbol{\phi},\boldsymbol{\xi},\lambda_0 \right) \right\} \Pi(\mathcal{B}) \\
&\geq \exp\left[-C_1 \max\left\{ \lambda^2 s \|\mathbf{B}_0\|_{2\to\infty}^2, sr \left| \log \frac{\lambda\eta}{\sqrt{rs}} \right|, s\log p \right\} \right],
\end{aligned}
$$

where $C_1 > 0$ is some absolute constant. □

Proof of Lemma 3. Recall that by construction, $(\|\mathbf{B}_{j*}\|_1 \mid \xi_j = 1) \sim \mathrm{Gamma}(r,\lambda)$ and $(\|\mathbf{B}_{j*}\|_1 \mid \xi_j = 0, \lambda_0) \sim \mathrm{Exp}(\lambda_0+\lambda)$, and $(\xi_j \mid \theta) \sim \mathrm{Bernoulli}(\theta)$ independently for each $j \in [p]$. Then with $\boldsymbol{\xi}$ integrated out, we have, independently for each $j \in [p]$,

$$
\pi(\|\mathbf{B}_{j*}\|_1 \mid \theta, \lambda_0) = (1-\theta)(\lambda_0+\lambda)\mathrm{e}^{-(\lambda_0+\lambda)\|\mathbf{B}_{j*}\|_1} + \theta\frac{\lambda^r}{\Gamma(r)}\|\mathbf{B}_{j*}\|_1^{r-1}\mathrm{e}^{-\lambda\|\mathbf{B}_{j*}\|_1}.
$$

Therefore, with λ_0 integrated out, for any $\delta > 1/p^\gamma$, we obtain

$$
\begin{aligned}
\Pi(\|\mathbf{B}_{j*}\|_1 > \delta \mid \theta) &\leq (1-\theta)\frac{1}{\Gamma(1/p^2)}\int_0^\infty \lambda_0^{-1/p^2-1}\mathrm{e}^{-1/\lambda_0}\mathrm{e}^{-(\lambda_0+\lambda)\delta}\mathrm{d}\lambda_0 + \theta \\
&\leq \frac{2}{\mathrm{e}^p}\int_0^\infty u^{1/p^2-1}\exp\left(-\frac{\delta}{u}-u\right)\mathrm{d}u + \theta,
\end{aligned}
$$

where the last inequality is due to the change of variable $u = 1/\lambda_0$ and the fact that $\Gamma(1/p^2) \geq \mathrm{e}^p/2$ for sufficiently large p. Now we break down the integral in the preceding display as follows:

$$
\int_0^\infty u^{1/p^2-1}\exp\left(-\frac{\delta}{u}-u\right)\mathrm{d}u \leq \int_0^{4\delta} u^{1/p^2-1}\exp\left(-\frac{\delta}{u}\right)\mathrm{d}u + \int_{4\delta}^\infty u^{1/p^2-1}\exp\left(-u\right)\mathrm{d}u.
$$

For the first term, we observe that the function $u \mapsto (1/p^2-1)\log u - \delta/u$ achieves the maximum at $u = \delta/(1-p^{-2})$, and therefore, for sufficiently large p (small δ)

$$
\int_0^{4\delta} u^{1/p^2-1}\exp\left(-\frac{\delta}{u}\right)\mathrm{d}u \leq 4\delta\exp\left\{\left(1-\frac{1}{\mathrm{e}^p}\right)\left(\log\frac{1-p^{-2}}{\delta}\right)\right\} \leq 4\delta^{1/p^2} \leq \log\frac{1}{\delta}.
$$

For the second term, we apply the technique developed by [42] to derive

$$
\int_{4\delta}^\infty u^{1/p^2-1}\exp\left(-u\right)\mathrm{d}u \leq \int_{4\delta}^\infty u^{-1}\mathrm{e}^{-u/2}\mathrm{d}u \leq \log\frac{1}{\delta},
$$

where the inequality for incomplete Gamma function due to [80] is applied. Therefore, for any θ in the event $\{\theta < A_1 s\log p/p^{1+\kappa}\}$ for some constant A_1 to be determined later, we obtain

$$\Pi(\|\mathbf{B}_{j*}\|_1 > \delta \mid \theta) \leq \frac{4}{\mathrm{e}^p}\left(\log\frac{1}{\delta}\right) + \theta \leq \frac{4\gamma\log p + A_1 s\log p}{p^{1+\kappa}} \leq \frac{s\log p}{p}\left(\frac{A_1 + 4\gamma}{p^\kappa}\right).$$

A version of the Chernoff's inequality for binomial distributions states that [81]

$$\mathbb{P}(X > ap) \leq \left\{\left(\frac{q}{a}\right)^a \exp(a)\right\}^p \quad \text{if } X \sim \text{Binomial}(p, q) \text{ and } q \leq a < 1.$$

Then over the event $\{\theta < A_1 s\log p/p^{1+\kappa}\}$, we have

$$\begin{aligned}\Pi(|\mathrm{supp}_\delta(\mathbf{B})| > \beta s \mid \theta) &\leq \exp\left[-\beta s\left\{\log\frac{\beta}{\mathrm{e}(A_1 + 4\gamma)\log p} + \kappa\log p\right\}\right] \\ &= \exp\left(-\frac{1}{2}\beta\kappa s\log p\right)\end{aligned}$$

by taking $A_1 = \beta/\mathrm{e} - 4\gamma$, $q = \Pi(\|\mathbf{B}_{j*}\|_1 > \delta \mid \theta) \leq (A_1 + 4\gamma)s\log p/p^{1+\kappa}$, and $a = \beta s/p$. Observe that for sufficiently small x, $(1-x)^{1/x} \leq \mathrm{e}^{-1/2}$. Then we integrate with respect to $\Pi(\mathrm{d}\theta)$ and proceed to compute

$$\begin{aligned}\Pi(|\mathrm{supp}_\delta(\mathbf{B})| > \beta s) &= \int_0^1 \Pi(|\mathrm{supp}_\delta(\mathbf{B})| > \beta s \mid \theta)\Pi(\mathrm{d}\theta) \\ &\leq \int_0^{A_1 s\log p/p^{1+\kappa}} \Pi(|\mathrm{supp}_\delta(\mathbf{B})| > \beta s \mid \theta)\Pi(\mathrm{d}\theta) \qquad (5.1) \\ &\quad + \Pi\left(\theta > \frac{A_1 s\log p}{p^{1+\kappa}}\right) \\ &\leq \exp\left(-\frac{1}{2}\beta\kappa s\log p\right) + \left\{\left(1 - \frac{A_1 s\log p}{p^{1+\kappa}}\right)^{p^{1+\kappa}/(A_1 s\log p)}\right\}^{A_1 s\log p} \\ &\leq \exp\left(-\frac{1}{2}\beta\kappa s\log p\right) + \exp\left(-\frac{A_1}{2}s\log p\right) \\ &\leq 2\exp\left\{-\min\left(\frac{\beta\kappa}{2}, \frac{\beta}{2\mathrm{e}} - 2\gamma\right)s\log p\right\},\end{aligned}$$

and the proof is thus completed. □

Proof of Lemma 4. To proof Lemma 4, we need the following technical results regarding moments of Gamma mixture distributions, the proof of which is deferred to Section 5.1.5.

Lemma 7 *Suppose that w follows a mixture of an exponential distribution* $\mathrm{Exp}(\lambda_0)$ *and a Gamma distribution* $\mathrm{Gamma}(r, \lambda)$*, with mixing weights $1-\theta$ and θ, respectively. Let $\xi = \mathbb{1}(w > \delta)$, where δ is some sufficiently small constant such that $\Gamma(r) \le 2\Gamma(r, \lambda\delta)$, and $\Gamma(r,\delta) = \int_\delta^\infty w^{r-1}\mathrm{e}^{-w}\mathrm{d}w$ is the (upper) incomplete Gamma function. Then the moments of w satisfy*

$$\sup_{m\ge 1}\frac{1}{m}\left\{E(w^m \mid \xi = 1)\right\}^{1/m} \le 2\delta + \frac{2}{\lambda_0} + \frac{2(r+1)}{\lambda} \text{ and } \sup_{m\ge 1}\frac{1}{m}\{E(w^m)\}^{1/m} \le \frac{1}{\lambda_0} + \frac{r+1}{\lambda}.$$

Furthermore, if $\theta \le \mathrm{e}^{-r}$, then the moments of w satisfies

$$\sup_{m\ge 1}\frac{1}{m}\{E(w^m)\}^{1/m} \le \frac{1}{\lambda_0} + \frac{1}{\lambda}.$$

Denote $S_0 = \mathrm{supp}(\mathbf{B}_0)$. We first use the union bound to derive

$$\begin{aligned}
&\Pi\left[\sum_{j=1}^p \|\mathbf{B}_{j*}\|_1 \mathbb{1}\{j \in \mathrm{supp}_{\delta_n}(\mathbf{B}) \cup \mathrm{supp}(\mathbf{B}_0)\} \ge t_n\right] \\
&\quad \le \Pi\left\{\sum_{j=1}^p \|\mathbf{B}_{j*}\|_1 \mathbb{1}(\|\mathbf{B}_{j*}\|_1 > \delta_n) \ge t_n/2\right\} + \Pi\left(\sum_{j\in S_0}\|\mathbf{B}_{j*}\|_1 \ge t_n/2\right),
\end{aligned}$$

and then analyze the above two terms separately.

Upper bounding the second term. Recall that

$$\pi(\|\mathbf{B}_{j*}\|_1 \mid \lambda_0, \theta) = (1-\theta)(\lambda_0+\lambda)\mathrm{e}^{-(\lambda_0+\lambda)\|\mathbf{B}_{j*}\|_1} + \theta\frac{\lambda^r}{\Gamma(r)}\|\mathbf{B}_{j*}\|^{r-1}\mathrm{e}^{-\lambda\|\mathbf{B}_{j*}\|_1}.$$

Denote $\beta' = \beta/\mathrm{e} - 4\gamma > 0$. Over the event $\{\theta \le (\beta' s \log p)/p^{1+\kappa}\}$, it holds that

$$\mathbb{E}_\Pi(\|\mathbf{B}_{j*}\|_1 \mid \theta, \lambda_0) \le \frac{1}{\lambda_0} + \frac{\beta' s\log p}{\lambda p^{1+\kappa}} \le \frac{2}{\lambda}.$$

Since $(\beta' s \log p)/p^{1+\kappa} \le 1/\sqrt{p} = \mathrm{e}^{-(\log p)/2} \le \mathrm{e}^{-r}$ for sufficiently large n, we invoke Lemma 7 to derive

$$\sup_{m\ge 1}\{\mathbb{E}_\Pi\left(\|\mathbf{B}_{j*}\|_1^m \mid \theta, \lambda_0\right)\}^{1/m} \le \frac{2}{\lambda}$$

over the event $\{\theta \le (\beta' s \log p)/p^{1+\kappa}\}$, and proceed to apply the large deviation inequality for sub-exponential random variables (Proposition 5.16 in [82]) to obtain

$$\Pi\left(\sum_{j\in S_0}\|\mathbf{B}_{j*}\|_1 \ge t_n/2 \,\middle|\, \lambda_0, \theta\right)$$

$$
\begin{aligned}
&\leq \Pi\left[\sum_{j\in S_0}\{\|\mathbf{B}_{j*}\|_1 - \mathbb{E}_\Pi(\|\mathbf{B}_{j*}\|_1 \mid \lambda_0,\theta)\} \geq t_n/2 - \frac{2s}{\lambda}\,\middle|\, \lambda_0,\theta\right]\\
&\leq \Pi\left[\sum_{j\in S_0}\{\|\mathbf{B}_{j*}\|_1 - \mathbb{E}_\Pi(\|\mathbf{B}_{j*}\|_1 \mid \lambda_0,\theta)\} \geq t_n/4\,\middle|\, \lambda_0,\theta\right]\\
&\leq \exp\left\{-C\min\left(\frac{t_n^2}{s_n^2}, t_n\right)\right\}
\end{aligned}
$$

for sufficiently large n, where C is some absolute constant. Observe that

$$
\begin{aligned}
\Pi\left(\theta > \frac{\beta' s\log p}{p^{1+\kappa}}\right) &= \left\{\left(1 - \frac{\beta' s\log p}{p^{1+\kappa}}\right)^{p^{1+\kappa}/(\beta' s\log p)}\right\}^{\beta' s\log p}\\
&\leq \exp\left\{-\left(\frac{\beta}{2\mathrm{e}} - 4\gamma\right)s\log p\right\}
\end{aligned}
$$

since $(1-x)^{1/x} \leq \mathrm{e}^{-1/2}$ for $x \leq 1$, and so we obtain

$$
\begin{aligned}
\Pi\left(\sum_{j\in S_0}\|\mathbf{B}_{j*}\|_1 \geq t_n/2\right) &\leq \mathbb{E}_\Pi\left[\Pi\left(\sum_{j\in S_0}\|\mathbf{B}_{j*}\|_1 \geq t_n/2\,\middle|\,\lambda_0,\theta\right)\mathbb{1}\left(\theta \leq \frac{\beta' s\log p}{p^{1+\kappa}}\right)\right]\\
&\quad + \Pi\left(\theta \leq \frac{\beta' s\log p}{p^{1+\kappa}}\right)\\
&\leq \exp\left\{-C\min\left(\frac{t_n^2}{s_n^2}, t_n\right)\right\} + \exp\left\{-\left(\frac{\beta}{2\mathrm{e}} - 4\gamma\right)s\log p\right\}
\end{aligned}
$$

for sufficiently large n.

Upper bounding the first term. Denote $\zeta_j = \mathbb{1}(\|\mathbf{B}_{j*}\|_1 > \delta_n)$ and $\boldsymbol{\zeta} = [\zeta_1,\ldots,\zeta_p]^{\mathrm{T}}$. By Lemma 7 we obtain the following bound for the conditional expected value and moments of $\|\mathbf{B}_{j*}\|_1$ given $\zeta_j = 1$ and θ for sufficiently large n:

$$
\mathbb{E}_\Pi(\|\mathbf{B}_{j*}\|_1 \mid \zeta_j = 1,\theta) \leq \sup_{m\geq 1}\{\mathbb{E}_\Pi(\|\mathbf{B}_{j*}\|_1^m \mid \zeta_j = 1,\theta)\}^{1/m} \leq 2\delta_n + \frac{2}{\lambda_0} + \frac{2(r+1)}{\lambda} \leq \frac{8r}{\lambda}.
$$

Since $|\mathrm{supp}_{\delta_n}(\mathbf{B})| = \sum_{j=1}^p \zeta_j$, then over the event $\{\boldsymbol{\zeta} : |\mathrm{supp}_{\delta_n}(\mathbf{B})| \leq \beta s\}$, we invoke the large deviation inequality for sub-exponential random variables again to derive

$$
\begin{aligned}
&\Pi\left(\sum_{j=1}^p\|\mathbf{B}_{j*}\|_1\zeta_j > t_n/2\,\middle|\,\boldsymbol{\zeta},\theta\right)\\
&\quad\leq \Pi\left[\sum_{j\in\mathrm{supp}_{\delta_n}(\mathbf{B})}\{\|\mathbf{B}_{j*}\|_1 - \mathbb{E}_\Pi(\|\mathbf{B}_{j*}\|_1 \mid \zeta_j = 1)\} > \frac{t_n}{2} - \frac{8sr}{\lambda}\,\middle|\,\boldsymbol{\zeta},\theta\right]
\end{aligned}
$$

$$\leq \Pi\left[\sum_{j\in\mathrm{supp}_{\delta_n}(\mathbf{B})}\{\|\mathbf{B}_{j*}\|_1 - \mathbb{E}_\Pi(\|\mathbf{B}_{j*}\|_1 \mid \zeta_j = 1)\} > \frac{t_n}{4} \,\middle|\, \boldsymbol{\zeta},\theta\right]$$
$$\leq \exp\left[-C\min\left\{\left(\frac{t_n}{\beta sr}\right)^2, \frac{t_n}{r}\right\}\right]$$

for sufficiently large n. Invoking Lemma 3, we obtain

$$\Pi\left(\sum_{j=1}^p \|\mathbf{B}_{j*}\|_1\zeta_j > t_n/2\right)$$
$$\leq \mathbb{E}_\Pi\left\{\Pi\left(\sum_{j=1}^p \|\mathbf{B}_{j*}\|_1\zeta_j > t_n/2 \,\middle|\, \boldsymbol{\zeta},\theta\right)\mathbb{1}(\boldsymbol{\zeta} : |\mathrm{supp}_{\delta_n}(\mathbf{B})| \leq \beta s)\right\}$$
$$+ \Pi\left(|\mathrm{supp}_{\delta_n}(\mathbf{B})| \leq \beta s\right)$$
$$\leq \exp\left[-C\min\left\{\left(\frac{t_n}{\beta sr}\right)^2, \frac{t_n}{r}\right\}\right] + 2\exp\left\{-\min\left(\frac{\beta\kappa}{2}, \frac{\beta}{2\mathrm{e}} - 2\gamma\right)s\log p\right\}.$$

Combining upper bounds: Combining the previous two upper bounds, we obtain

$$\Pi\left[\sum_{j=1}^p \|\mathbf{B}_{j*}\|_1\mathbb{1}\{j \in \mathrm{supp}_{\delta_n}(\mathbf{B}) \cup \mathrm{supp}(\mathbf{B}_0)\} \geq t_n\right]$$
$$\leq 2\exp\left[-C\min\left\{\left(\frac{t_n}{\beta sr}\right)^2, \left(\frac{t_n}{s}\right)^2, \frac{t_n}{r}\right\}\right] + 3\exp\left\{-\min\left(\frac{\beta\kappa}{2}, \frac{\beta}{2\mathrm{e}} - 2\gamma\right)s\log p\right\},$$

and the proof is completed. □

5.1.2 Proofs of results in Section 2.4.2

Proof of Theorem 3. Recall that $\mathcal{U}_n = \{\|\boldsymbol{\Sigma} - \boldsymbol{\Sigma}_0\|_2 \leq M\epsilon_n\}$ and the posterior probability $\Pi(\mathcal{U}_n^c \mid \mathbf{Y}_n)$ can be written as $\Pi(\mathcal{U}_n^c \mid \mathbf{Y}_n) = N_n(\mathcal{U}_n^c)/D_n$, where

$$N_n(\mathcal{U}_n^c) = \int_{\mathcal{A}} \exp\{\ell_n(\boldsymbol{\Sigma}) - \ell_n(\boldsymbol{\Sigma}_0)\}\Pi(\mathrm{d}\boldsymbol{\Sigma}),\quad D_n = \int \exp\{\ell_n(\boldsymbol{\Sigma}) - \ell_n(\boldsymbol{\Sigma}_0)\}\Pi(\mathrm{d}\boldsymbol{\Sigma}),$$

and $\ell_n(\boldsymbol{\Sigma}) = \sum_{i=1}^n \log p(\mathbf{y}_i \mid \boldsymbol{\Sigma})$ is the log-likelihood function of $\boldsymbol{\Sigma}$.

Step 1: Prior concentration. Let $\eta_n = \sqrt{(s\log p)/n}$. Then by Lemma 5, there exists a sequence of events $(\mathcal{A}_n)_{n=1}^\infty$ such that

$$\mathcal{A}_n \subset \{D_n \geq \Pi(\|\boldsymbol{\Sigma} - \boldsymbol{\Sigma}_0\|_\mathrm{F} \leq \eta_n)\exp(-C_3' s\log p)\}$$

for $\eta_n = \sqrt{(s\log p)/n} \leq \sigma_0^2/2$, and

$$\mathbb{P}_0(\mathcal{A}_n^c) \leq 2\exp\left\{-\tilde{C}_3 \min\left(1, \|\boldsymbol{\Sigma}_0^{-1}\|_2^{-2}\right) s\log p\right\}, \tag{5.2}$$

where C_3' and $\tilde{C}_3$ are some absolute constants. Denote $\mathbf{B}_0 = \mathbf{U}_0\boldsymbol{\Lambda}_0^{1/2}$, where $\boldsymbol{\Lambda}_0^{1/2} = \mathrm{diag}(\lambda_{01}^{1/2}, \ldots, \lambda_{0r}^{1/2})$. Then we analyze the prior concentration using a union bound as follows:

$$\begin{aligned}
\Pi(\|\boldsymbol{\Sigma} - \boldsymbol{\Sigma}_0\|_{\mathrm{F}} \leq \eta_n) &\geq \Pi\left(\|\mathbf{B}\mathbf{B}^{\mathrm{T}} - \mathbf{B}_0\mathbf{B}_0^{\mathrm{T}}\|_{\mathrm{F}} + \|\sigma^2\mathbf{I}_p - \sigma_0^2\mathbf{I}_p\|_{\mathrm{F}} \leq \eta_n\right)\\
&\geq \Pi\left(\|\mathbf{B}\mathbf{B}^{\mathrm{T}} - \mathbf{B}_0\mathbf{B}_0^{\mathrm{T}}\|_{\mathrm{F}} \leq \frac{\eta_n}{2}\right)\Pi\left(|\sigma_0^2 - \sigma^2| \leq \frac{\eta_n}{2\sqrt{p}}\right).
\end{aligned}$$

On one hand, for $\eta_n = \sqrt{(s\log p)/n} \leq \sigma_0^2/2$, we have

$$\Pi\left(|\sigma_0^2 - \sigma^2| \leq \frac{\eta_n}{2\sqrt{p}}\right) \geq \left\{\min_{\sigma\in[\sigma_0^2/2, 3\sigma_0^2/2]} \pi_\sigma(\sigma^2)\right\}\frac{\eta_n}{\sqrt{p}} \geq C(\sigma_0^2)\mathrm{e}^{-\log p},$$

where the constant $C(\sigma_0^2) = \min_{\sigma_0^2/2\leq\sigma^2\leq 3\sigma_0^2/2}\pi_\sigma(\sigma^2) > 0$ depends only on σ_0^2. On the other hand, for $\eta_n = \sqrt{(s\log p)/n} \leq \min(\sigma_0^2/2, 16\|\mathbf{B}_0\|_2^{1/2})$, we proceed by union bound to derive

$$\begin{aligned}
&\Pi\left(\|\mathbf{B}\mathbf{B}^{\mathrm{T}} - \mathbf{B}_0\mathbf{B}_0^{\mathrm{T}}\|_{\mathrm{F}} \leq \frac{\eta_n}{2}\right)\\
&\quad\geq \Pi\left(\|\mathbf{B} - \mathbf{B}_0\|_{\mathrm{F}}\|\mathbf{B} - \mathbf{B}_0 + \mathbf{B}_0\|_2 + \|\mathbf{B}_0\|_2\|\mathbf{B} - \mathbf{B}_0^{\mathrm{T}}\|_{\mathrm{F}} \leq \frac{\eta_n}{2}\right)\\
&\quad\geq \Pi\left\{\|\mathbf{B} - \mathbf{B}_0\|_{\mathrm{F}}\left(\|\mathbf{B} - \mathbf{B}_0\|_{\mathrm{F}} + 2\|\mathbf{B}_0\|_2\right) \leq \frac{\eta_n}{2}\right\}\\
&\quad\geq \Pi\left\{\|\mathbf{B} - \mathbf{B}_0\|_{\mathrm{F}} \leq \min\left(\frac{\eta_n}{8\|\mathbf{B}_0\|_2}, 2\|\mathbf{B}_0\|_2\right)\right\}\\
&\quad= \Pi\left(\|\mathbf{B} - \mathbf{B}_0\|_{\mathrm{F}} \leq \frac{\eta_n}{8\|\mathbf{B}_0\|_2}\right).
\end{aligned}$$

Invoking Lemma 2, we see that there exists some constant $C(\lambda, \mathbf{B}_0)$ depending on λ and $\|\mathbf{B}_0\|_{2\to\infty}$ only, such that

$$\begin{aligned}
&\Pi\left(\|\mathbf{B}\mathbf{B}^{\mathrm{T}} - \mathbf{B}_0\mathbf{B}_0^{\mathrm{T}}\|_{\mathrm{F}} \leq \frac{\eta_n}{2}\right)\\
&\quad\geq \Pi\left(\|\mathbf{B} - \mathbf{B}_0\|_{\mathrm{F}} \leq \frac{\eta_n}{8\|\mathbf{B}_0\|_2}\right)
\end{aligned}$$

$$
\begin{aligned}
&\geq \exp\left[-C_1 \max\left\{\lambda^2 s\|\mathbf{B}_0\|_{2\to\infty}^2, s\log p, sr\left|\log\left(\lambda\frac{\sqrt{\log p}}{\sqrt{rn}}\right)\right|\right\}\right]\\
&\geq \exp\left\{-C(\lambda,\mathbf{B}_0)s\log p\right\}.
\end{aligned}
$$

Therefore, for $\eta_n = \sqrt{(s\log p)/n} \leq \min(\sigma_0^2, 16\|\mathbf{B}_0\|_2^{1/2})$ we obtain

$$
\Pi(\|\boldsymbol{\Sigma} - \boldsymbol{\Sigma}_0\|_{\mathrm{F}} \leq \eta_n) \geq C(\sigma_0^2)\exp\left[-\{1 + C(\lambda,\mathbf{B}_0)\}s\log p\right],
$$

and over $\mathcal{A}_n$, we have

$$
D_n \geq C(\sigma_0^2)\exp\left(-C_{0\lambda}s\log p\right) \tag{5.3}
$$

for some constant $C_{0\lambda}$ depending only on λ and $\|\mathbf{B}_0\|_{2\to\infty}$.

Step 2: Construct subsets $(\mathcal{F}_n)_{n=1}^\infty$. Take $\epsilon_n = \sqrt{(s\log p)/n}$, $\tau_n = \beta s_n$, $t_n = (sr\log p)^2$, and $\delta_n = \epsilon_n/(t_n\sqrt{p})$, where $\beta > 0$ is some constant to be specified later. Clearly, there exists some $\gamma > 0$ such that

$$
\delta_n = \frac{\epsilon_n}{t_n\sqrt{p}} = \frac{\sqrt{s\log p}}{\sqrt{np}(sr\log p)^2} = \frac{1}{\sqrt{nps^3r^4(\log p)^3}} \geq \frac{1}{p^\gamma}.
$$

Now let $\beta > 4\mathrm{e}\gamma$ and $\mathcal{F}_n = \mathcal{F}(\delta_n, \tau_n, t_n)$ be defined in Lemma 6. Since

$$
\begin{aligned}
\min\left\{\left(\frac{t_n}{\beta sr}\right)^2, \left(\frac{t_n}{r}\right)^2, \frac{t_n}{r}\right\} &= \min\left\{\frac{(sr)^2(\log p)^4}{\beta^2}, s^4r^2(\log p)^4, s^2r(\log p)\right\}\\
&= \min\left\{\frac{sr^2(\log p)^3}{\beta^2}, s^3r^2(\log p)^3, sr\log p\right\} s\log p\\
&\geq \beta s\log p
\end{aligned}
$$

for sufficiently large n, and $t_n/(sr) = (sr)\log p \to \infty$, we then can invoke Lemmas 3 and 4 to obtain

$$
\begin{aligned}
\Pi(\mathcal{F}_n^c) &\leq \Pi(|\mathrm{supp}_{\delta_n}(\mathbf{B})| > \beta s_n) + \Pi\left[\sum_{j=1}^p \|\mathbf{B}_{j*}\|_2^2 \mathbb{1}\{j \in \mathrm{supp}_{\delta_n}(\mathbf{B}) \cup \mathrm{supp}(\mathbf{U}_0)\} > t_n^2\right]\\
&\leq 2\exp(-\beta s\log p) + 5\exp\left\{-\min\left(\frac{\beta\kappa}{2}, \frac{\beta}{2\mathrm{e}} - 2\gamma\right)s\log p\right\}\\
&\leq 7\exp\left\{-\min\left(\frac{\beta\kappa}{2}, \frac{\beta}{2\mathrm{e}} - 2\gamma\right)s\log p\right\}
\end{aligned} \tag{5.4}
$$

for sufficiently large n (and hence sufficiently small $s\log p/p$).

Step 3: Decompose the integral $\mathbb{E}_0\{\Pi(\mathcal{U}_n^c \mid \mathbf{Y}_n)\}$. Since by construction we have

$$(\sqrt{p}\delta_n + 2t_n)\sqrt{p}\delta_n = \left(\sqrt{p}\frac{\epsilon_n}{t_n\sqrt{p}} + 2t_n\right)\sqrt{p}\delta_n \leq 3t_n\sqrt{p}\delta_n = 3\epsilon_n.$$

Then by Lemma 6, for each $M \geq \max\{3/2, (128\|\mathbf{\Sigma}_0\|_2^4)^{1/3}\}$, there exists a test function ϕ_n such that

$$\mathbb{E}_0(\phi_n) \leq 3\exp\left[-\left\{\frac{C_4\sqrt{M}}{\sqrt{2}} - (2+C_4)(\beta+2)\right\}s\log p\right], \tag{5.5}$$

$$\sup_{\mathbf{\Sigma}\in\mathcal{U}_n^c\cap\mathcal{F}_n} \mathbb{E}_{\mathbf{\Sigma}}(1-\phi_n) \leq \exp\left[-\left\{\frac{C_4 M}{8} - C_4(\beta+2)\right\}s\log p\right] \tag{5.6}$$

for some absolute constant $C_4 > 0$ for sufficiently large n. Now we decompose the target integral $\mathbb{E}_0\{\Pi(\mathcal{U}_n^c \mid \mathbf{Y}_n)\}$ using (5.2) and (5.5) as follows:

$$\begin{aligned}
\mathbb{E}_0\{\Pi(\mathcal{U}_n^c \mid \mathbf{Y}_n)\} &\leq \mathbb{E}_0(\phi_n) + \mathbb{E}_0\left\{(1-\phi_n)\Pi(\mathcal{U}_n \mid \mathbf{Y}_n)\mathbb{1}(\mathcal{A}_n)\right\} + \mathbb{P}_0(\mathcal{A}_n^c)\\
&\leq 3\exp\left[-\left\{\frac{C_4\sqrt{M}}{\sqrt{2}} - (2+C_4)(\beta+2)\right\}s\log p\right]\\
&\quad + 2\exp\left\{-\tilde{C}_3\min\left(1, \|\mathbf{\Sigma}_0^{-1}\|_2^{-2}\right)s\log p\right\}\\
&\quad + \mathbb{E}_0\left[(1-\phi_n)\left\{\frac{N_n(\mathcal{U}_n^c)}{D_n}\right\}\mathbb{1}(\mathcal{A}_n)\right].
\end{aligned}$$

Now we focus on the third term on the right-hand side of the preceding display. By (5.3), we obtain

$$\begin{aligned}
&\mathbb{E}_0\left[(1-\phi_n)\left\{\frac{N_n(\mathcal{U}_n^c)}{D_n}\right\}\mathbb{1}(\mathcal{A}_n)\right]\\
&\quad\leq \frac{\exp\left(C_{0\lambda}s\log p\right)}{C(\sigma_0^2)}\mathbb{E}_0\left\{(1-\phi_n)\int_{\mathcal{U}_n^c}\prod_{i=1}^n\frac{p(\mathbf{y}_i \mid \mathbf{\Sigma})}{p(\mathbf{y}_i \mid \mathbf{\Sigma}_0)}\Pi(\mathrm{d}\mathbf{\Sigma})\right\}.
\end{aligned}$$

Observe that by Fubini's theorem,

$$\begin{aligned}
&\mathbb{E}_0\left\{(1-\phi_n)\int_{\mathcal{U}_n^c}\prod_{i=1}^n\frac{p(\mathbf{y}_i \mid \mathbf{\Sigma})}{p(\mathbf{y}_i \mid \mathbf{\Sigma}_0)}\Pi(\mathrm{d}\mathbf{\Sigma})\right\}\\
&\leq \mathbb{E}_0\left\{(1-\phi_n)\int_{\mathcal{U}_n^c\cap\mathcal{F}_n}\prod_{i=1}^n\frac{p(\mathbf{y}_i \mid \mathbf{\Sigma})}{p(\mathbf{y}_i \mid \mathbf{\Sigma}_0)}\Pi(\mathrm{d}\mathbf{\Sigma})\right\} + \mathbb{E}_0\left\{\int_{\mathcal{F}_n^c}\prod_{i=1}^n\frac{p(\mathbf{y}_i \mid \mathbf{\Sigma})}{p(\mathbf{y}_i \mid \mathbf{\Sigma}_0)}\Pi(\mathrm{d}\mathbf{\Sigma})\right\}\\
&= \int_{\mathcal{U}_n^c\cap\mathcal{F}_n}\mathbb{E}_0\left\{(1-\phi_n)\prod_{i=1}^n\frac{p(\mathbf{y}_i \mid \mathbf{\Sigma})}{p(\mathbf{y}_i \mid \mathbf{\Sigma}_0)}\right\}\Pi(\mathrm{d}\mathbf{\Sigma}) + \int_{\mathcal{F}_n^c}\left\{\mathbb{E}_0\prod_{i=1}^n\frac{p(\mathbf{y}_i \mid \mathbf{\Sigma})}{p(\mathbf{y}_i \mid \mathbf{\Sigma}_0)}\right\}\Pi(\mathrm{d}\mathbf{\Sigma})
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_{\mathcal{U}_n^c \cap \mathcal{F}_n} \mathbb{E}_{\boldsymbol{\Sigma}}(1-\phi_n)\Pi(\mathrm{d}\boldsymbol{\Sigma}) + \Pi(\mathcal{F}_n^c) \\
&\leq \exp\left[-\left\{\frac{C_4 M}{8} - C_4(\beta+2)\right\} s\log p\right] + 7\exp\left\{-\min\left(\frac{\beta\kappa}{2}, \frac{\beta}{2\mathrm{e}} - 2\gamma\right) s\log p\right\},
\end{aligned}
$$

where the testing type II error probability bound (5.6) and (5.4) are applied to the last inequality. Then by taking

$$
\begin{aligned}
\beta &= \max\left\{\frac{4}{\kappa}C_{0\lambda}, 2\mathrm{e}\left(2\gamma + 2C_{0\lambda}\right)\right\}, \\
M = M_0 &= \max\left[\frac{8}{C_4}\left\{C_4(\beta+2) + 2C_{0\lambda}\right\}, \frac{2}{C_4^2}\left\{C_{0\lambda} + (2+C_4)(\beta+2)\right\}^2\right],
\end{aligned}
$$

we obtain the following result:

$$
\begin{aligned}
&\mathbb{E}_0\left[(1-\phi_n)\left\{\frac{N_n(\mathcal{U}_n^c)}{D_n}\right\}\mathbb{1}(\mathcal{A}_n)\right] \\
&\quad\leq \frac{1}{C(\sigma_0^2)}\exp\left[-\left\{\frac{C_4 M}{8} - C_4(\beta+2) - C_{0\lambda}\right\} s\log p\right] \\
&\quad\quad+ \frac{1}{C(\sigma_0^2)} 7\exp\left[-\left\{\min\left(\frac{\beta\kappa}{2}, \frac{\beta}{2\mathrm{e}} - 2\gamma\right) - C_{0\lambda}\right\} s\log p\right] \\
&\quad\leq \frac{8}{C(\sigma_0^2)}\exp\left\{-C_{0\lambda} s\log p\right\}.
\end{aligned}
$$

Combining the above results, we finally obtain

$$
\begin{aligned}
\mathbb{E}_0\{\Pi(\mathcal{U}_n^c \mid \mathbf{Y}_n)\} &\leq \left\{3 + \frac{8}{C(\sigma_0^2)}\right\}\exp\left\{-C_{0\lambda} s\log p\right\} \\
&\quad + 2\exp\left\{-\tilde{C}_3\min\left(1, \left\|\boldsymbol{\Sigma}_0^{-1}\right\|_2^{-2}\right) s\log p\right\} \\
&\leq \left\{5 + \frac{11}{C(\sigma_0^2)}\right\}\exp\left[-\min\left\{C_{0\lambda}, \tilde{C}_3, \tilde{C}_3\|\boldsymbol{\Sigma}_0^{-1}\|_2^{-2}\right\} s\log p\right] \\
&= R_0\exp(-C_0 s\log p)
\end{aligned}
$$

by taking $C_0 = \min\left\{C_{0\lambda}, \tilde{C}_3, \tilde{C}_3\|\boldsymbol{\Sigma}_0^{-1}\|_2^{-2}\right\}$ and $R_0 = \{5 + 11/C(\sigma_0^2)\}$. Therefore, there exists some constant M_0, such that for all sufficiently large n, we have

$$
\begin{aligned}
\mathbb{E}_0\left\{\Pi\left(\|\boldsymbol{\Sigma} - \boldsymbol{\Sigma}_0\|_2 > M\epsilon_n \,\middle|\, \mathbf{Y}_n\right)\right\} &\leq \mathbb{E}_0\left\{\Pi\left(\|\boldsymbol{\Sigma} - \boldsymbol{\Sigma}_0\|_2 > M_0\epsilon_n \,\middle|\, \mathbf{Y}_n\right)\right\} \\
&\leq R_0\exp(-C_0 s\log p)
\end{aligned}
$$

for some absolute constants C_0 and R_0 depending on $\mathbf{\Sigma}_0$ and the hyperparameters only.

Step 4: Bounding the projection operator norm loss using the sine-theta theorem. To prove the posterior contraction for $\mathbf{U}$ with respect to the projection operator norm loss (2.7), we need the following version of the Davis-Kahan sine-theta theorem, which follows as a recasting of Theorem VII.3.7 in [83] in the language of [84]:

Theorem 21 *Let $\mathbf{X}, \widehat{\mathbf{X}} \in \mathbb{R}^{p\times p}$ be symmetric matrices with eigenvalues $\lambda_1 \geq \ldots \geq \lambda_p$ and $\hat{\lambda}_1 \geq \ldots \geq \hat{\lambda}_p$, respectively. Write $\mathbf{E} = \widehat{\mathbf{X}} - \mathbf{X}$ and fix $1 \leq r \leq s \leq p$. Assume that $\delta_{\mathrm{gap}} := \min(\lambda_{r-1} - \lambda_r, \lambda_s - \lambda_{s+1}) > 0$ where $\lambda_0 := \infty$ and $\lambda_{p+1} := -\infty$. Let $d = s - r + 1$ and let $\mathbf{V} = [\mathbf{v}_1, \ldots, \mathbf{v}_s] \in \mathbb{R}^{p\times d}$ and $\widehat{\mathbf{V}} = [\hat{\mathbf{v}}_r, \ldots, \hat{\mathbf{v}}_s] \in \mathbb{R}^{p\times d}$ have orthonormal columns satisfying $\mathbf{X}\mathbf{v}_j = \lambda_j \mathbf{v}_j$ and $\widehat{\mathbf{X}}\hat{\mathbf{v}}_j = \hat{\lambda}_j \hat{\mathbf{v}}_j$ for $j = r, r+1, \ldots, s$. Then*

$$\|\widehat{\mathbf{V}}\widehat{\mathbf{V}}^{\mathrm{T}} - \mathbf{V}\mathbf{V}^{\mathrm{T}}\|_2 \leq \frac{2}{\delta_{\mathrm{gap}}}\|\mathbf{E}\|_2.$$

To apply the sine-theta theorem, we let $\mathbf{X} = \mathbf{\Sigma}_0 = \mathbf{U}_0\mathbf{\Lambda}_0\mathbf{U}_0^{\mathrm{T}} + \sigma_0^2\mathbf{I}_p$, $\widehat{\mathbf{X}} = \mathbf{B}\mathbf{B}^{\mathrm{T}} + \sigma^2\mathbf{I}_p$, and take "$s$"$= r$ and "r"$= 1$, in which case $\delta_{\mathrm{gap}} = \min\{\infty, \lambda_r(\mathbf{\Sigma}_0) - \lambda_{r+1}(\mathbf{\Sigma}_0)\} = \lambda_{0r}$, $\mathbf{V} = \mathbf{U}_0$, $\widehat{\mathbf{V}} = \mathbf{U}_{\mathbf{B}}$, and $\mathbf{E} = \mathbf{\Sigma} - \mathbf{\Sigma}_0$. Then by the sine-theta theorem and (2.6), we have

$$\|\mathbf{U}_{\mathbf{B}}\mathbf{U}_{\mathbf{B}}^{\mathrm{T}} - \mathbf{U}_0\mathbf{U}_0^{\mathrm{T}}\|_2 \leq \frac{2}{\lambda_{0r}}\|\mathbf{E}\|_2 = \frac{2}{\lambda_{0r}}\|\mathbf{\Sigma} - \mathbf{\Sigma}_0\|_2$$

and hence, by the posterior contraction for $\mathbf{\Sigma}$, we have

$$\begin{aligned}\mathbb{E}_0\left\{\Pi\left(\|\mathbf{U}_{\mathbf{B}}\mathbf{U}_{\mathbf{B}}^{\mathrm{T}} - \mathbf{U}_0\mathbf{U}_0^{\mathrm{T}}\|_2 > \frac{2M\epsilon_n}{\lambda_{0r}}\,\middle|\,\mathbf{Y}_n\right)\right\} &\leq \mathbb{E}_0\left\{\Pi\left(\|\mathbf{\Sigma} - \mathbf{\Sigma}_0\|_2 > M\epsilon_n\,\middle|\,\mathbf{Y}_n\right)\right\}\\ &\leq R_0\exp(-C_0 s\log p).\end{aligned}$$

□

Proof of Theorem 4. The proof is similar to that of Theorem 3, but we need the following testing lemma dealing with the infinity norm loss $\|\mathbf{\Sigma} - \mathbf{\Sigma}_0\|_\infty$, which is analogous to Lemma 6. The proof is deferred to Section 5.1.4.

Lemma 8 *Assume the data* $\mathbf{y}_1, \ldots, \mathbf{y}_n$ *follows* $\mathrm{N}_p(\mathbf{0}_p, \mathbf{\Sigma})$, $1 \leq r \leq p$. *Suppose* $\mathbf{U}_0 \in \mathbb{O}(p, r)$ *satisfy* $|\mathrm{supp}(\mathbf{U}_0)| \leq s$, *and* $r \leq s \leq p$. *For any positive* δ, t, *and* τ, *define*

$$\mathcal{G}(\delta, \tau, t) = \left\{ \mathbf{B} \in \mathbb{R}^{p\times r} : |\mathrm{supp}_\delta(\mathbf{B})| \leq \tau, \sum_{j=1}^{p} \|\mathbf{B}_{j*}\|_1 \mathbb{1}\{j \in \mathrm{supp}_\delta(\mathbf{B}) \cup \mathrm{supp}(\mathbf{U}_0)\} \leq t \right\}.$$

Let the positive sequences $(\delta_n, \tau_n, t_n, \epsilon_n)_{n=1}^{\infty}$ *satisfy* $\max(p\delta_n t_n, \delta_n t_n + p\delta_n^2) \leq M_1 \epsilon_n$ *for some constant* $M_1 > 0$, *and* $\epsilon_n \leq 1$. *Consider testing* $H_0 : \mathbf{\Sigma} = \mathbf{\Sigma}_0 = \mathbf{U}_0 \mathbf{\Lambda}_0 \mathbf{U}_0^{\mathrm{T}} + \sigma_0^2 \mathbf{I}_p$ *versus*

$$H_1 : \mathbf{\Sigma} \in \left\{ \mathbf{\Sigma} = \mathbf{B}\mathbf{B}^{\mathrm{T}} + \sigma^2 \mathbf{I}_p : \|\mathbf{\Sigma} - \mathbf{\Sigma}_0\|_\infty > M\epsilon_n, \mathbf{B} \in \mathcal{G}(\delta_n, \tau_n, t_n) \right\}.$$

Then there exists some absolute constant $C_6 > 0$, *such that for each*

$$M \in \left[\max\left\{ \frac{M_1}{2}, 8, \frac{8(\log 2)^2}{C_6^2} \right\}, \frac{2\min(1, 2\|\mathbf{\Sigma}_0\|_2)}{\epsilon_n} \right],$$

there exists a test function $\phi_n : \mathbb{R}^{n\times p} \to [0, 1]$, *such that*

$$\mathbb{E}_0(\phi_n) \leq 12 \exp\left\{ 6(\tau_n \log p + 2s_n) - C_6 \min\left(\frac{1}{2}, \frac{\|\mathbf{\Sigma}_0\|_\infty^2}{\sqrt{2}} \right) \frac{\sqrt{M} n\epsilon_n^2}{\|\mathbf{\Sigma}_0\|_\infty^2} \right\},$$
$$\sup_{\mathbf{\Sigma}\in H_1} \mathbb{E}_{\mathbf{\Sigma}}(1 - \phi_n) \leq 4 \exp\left\{ 4(\tau_n + 2s_n) - C_6 \min\left(\frac{\|\mathbf{\Sigma}_0\|_\infty^2}{8}, \frac{1}{32} \right) \frac{M n\epsilon_n^2}{\|\mathbf{\Sigma}_0\|_\infty^2} \right\}.$$

Before we proceed to the proof, observe that the bounded coherence assumption on $\mathbf{U}_0$ (*i.e.*, $\|\mathbf{U}_0\|_{2\to\infty} \leq C_\mu \sqrt{r/s}$ for some $C_\mu \geq 1$) implies the following bound for the infinity norm on $\mathbf{\Sigma}_0$:

$$\begin{aligned}
\|\mathbf{\Sigma}_0\|_\infty &\leq \|\mathbf{U}_0 \mathbf{\Lambda}_0 \mathbf{U}_0^{\mathrm{T}}\|_\infty + \sigma_0^2 \leq \lambda_{01} \|\mathbf{U}_0\|_\infty \|\mathbf{U}_0^{\mathrm{T}}\|_\infty + \sigma_0^2 \\
&\leq \lambda_{01} \left(\sqrt{r} \|\mathbf{U}_0\|_{2\to\infty} \right) \left(\sqrt{s} \|\mathbf{U}_0^{\mathrm{T}}\|_{2\to\infty} \right) + \sigma_0^2 \leq C_\mu r \|\mathbf{\Sigma}_0\|_2.
\end{aligned}$$

Hence,

$$\frac{n\epsilon_n^2}{\|\mathbf{\Sigma}_0\|_\infty^2} = \frac{r^2 s \log p}{C_\mu^2 r^2 \|\mathbf{\Sigma}_0\|_2^2} = \frac{s \log p}{C_\mu^2 \|\mathbf{\Sigma}_0\|_2^2}.$$

Step 1 remains the same as that in the proof of Theorem 3. In what follows we will make use of inequalities (5.2) and (5.3).

Step 2: Construct subsets $(\mathcal{G}_n)_{n=1}^{\infty}$. This step is also similar to that in the proof of Theorem 3. Take $\epsilon_n = r\sqrt{(s\log p)/n}$, $\tau_n = \beta s_n$, $t_n = (sr\log p)^2$, and $\delta_n = \epsilon_n/(pt_n)$, where $\beta > 0$ is some constant to be specified later. Clearly, there exists some $\gamma > 0$ such that

$$\delta_n = \frac{\epsilon_n}{pt_n} = \frac{r\sqrt{s\log p}}{p\sqrt{n}(sr\log p)^2} = \frac{1}{\sqrt{np^2s^3r^2(\log p)^3}} \geq \frac{1}{p^\gamma}.$$

Now let $\beta > 4\mathrm{e}\gamma$ and $\mathcal{G}_n = \mathcal{G}(\delta_n, \tau_n, t_n)$ be defined in Lemma 8. Since

$$\min\left\{\left(\frac{t_n}{\beta sr}\right)^2, \left(\frac{t_n}{r}\right)^2, \frac{t_n}{r}\right\} \geq \beta s\log p$$

for sufficiently large n, and $t_n/(sr) = (sr)\log p \to \infty$, we then can invoke Lemmas 3 and 4 to obtain

$$\begin{aligned}
\Pi(\mathcal{G}_n^c) &\leq \Pi(|\mathrm{supp}_{\delta_n}(\mathbf{B})| > \beta s_n) + \Pi\left[\sum_{j=1}^p \|\mathbf{B}_{j*}\|_1 \mathbb{1}\{j \in \mathrm{supp}_{\delta_n}(\mathbf{B}) \cup \mathrm{supp}(\mathbf{U}_0)\} > t_n\right]\\
&\leq 2\exp(-\beta s\log p) + 5\exp\left\{-\min\left(\frac{\beta\kappa}{2}, \frac{\beta}{2\mathrm{e}} - 2\gamma\right)s\log p\right\}\\
&\leq 7\exp\left\{-\min\left(\frac{\beta\kappa}{2}, \frac{\beta}{2\mathrm{e}} - 2\gamma\right)s\log p\right\} \qquad (5.7)
\end{aligned}$$

for sufficiently large n (and hence sufficiently small $(s\log p)/p$).

Step 3: Decompose the integral. Since by construction we have

$$\max(p\delta_n t_n, \delta_n t_n + p\delta_n^2) \leq p\delta_n t_n + p\delta_n^2 \leq 2p\delta_n t_n \leq 2\epsilon_n,$$

then by Lemma 8, there exists some absolute constant $C_6 > 0$, such that for sufficiently large n, and for each

$$M \in \left[\max\left\{8, \frac{8(\log 2)^2}{C_6^2}\right\}, \frac{2\min(1, 2\|\mathbf{\Sigma}_0\|_2)}{\epsilon_n}\right],$$

there exists a test function ϕ_n such that

$$\mathbb{E}_0(\phi_n) \leq 12\exp\left[-\left\{\frac{C_6\sqrt{M}}{C_\mu^2\|\mathbf{\Sigma}_0\|_2^2}\min\left(\frac{1}{2}, \frac{\|\mathbf{\Sigma}_0\|_2^2}{\sqrt{2}}\right) - 6(\beta+2)\right\}s\log p\right], \qquad (5.8)$$

$$\mathbb{E}_{\boldsymbol{\Sigma}}(1-\phi_n) \le 4\exp\left[-\left\{\frac{C_6 M}{C_\mu^2\|\boldsymbol{\Sigma}_0\|_2^2}\min\left(\frac{\|\boldsymbol{\Sigma}_0\|_2^2}{8},\frac{1}{32}\right)-4(\beta+2)\right\}s\log p\right] \tag{5.9}$$

for all $\boldsymbol{\Sigma} \in \{\|\boldsymbol{\Sigma}-\boldsymbol{\Sigma}_0\|_\infty > M\epsilon_n\} \cap \mathcal{G}_n$. Denote $\mathcal{V}_n = \{\|\boldsymbol{\Sigma}-\boldsymbol{\Sigma}_0\|_\infty \le M\epsilon_n\}$. Now we decompose the target integral $\mathbb{E}_0\{\Pi(\mathcal{V}_n^c \mid \mathbf{Y}_n)\}$ using (5.2) and (5.8) as follows:

$$\begin{aligned}
\mathbb{E}_0\{\Pi(\mathcal{V}_n^c \mid \mathbf{Y}_n)\} &\le \mathbb{E}_0(\phi_n) + \mathbb{E}_0\left\{(1-\phi_n)\Pi(\mathcal{V}_n \mid \mathbf{Y}_n)\mathbb{1}(\mathcal{A}_n)\right\} + \mathbb{P}_0(\mathcal{A}_n^c)\\
&\le 12\exp\left[-\left\{\frac{C_6\sqrt{M}}{C_\mu^2\|\boldsymbol{\Sigma}_0\|_2^2}\min\left(\frac{1}{2},\frac{\|\boldsymbol{\Sigma}_0\|_2^2}{\sqrt{2}}\right)-6(\beta+2)\right\}s\log p\right]\\
&\quad + 2\exp\left\{-\tilde{C}_3\min\left(1,\|\boldsymbol{\Sigma}_0^{-1}\|_2^{-2}\right)s\log p\right\}\\
&\quad + \mathbb{E}_0\left[(1-\phi_n)\left\{\frac{N_n(\mathcal{V}_n^c)}{D_n}\right\}\mathbb{1}(\mathcal{A}_n)\right].
\end{aligned}$$

Now we focus on the third term on the right-hand side of the preceding display. By (5.3), we obtain

$$\begin{aligned}
&\mathbb{E}_0\left[(1-\phi_n)\left\{\frac{N_n(\mathcal{V}_n^c)}{D_n}\right\}\mathbb{1}(\mathcal{A}_n)\right]\\
&\quad\le \frac{\exp\{C_{0\lambda}\}s\log p}{C(\sigma_0^2)}\mathbb{E}_0\left\{(1-\phi_n)\int_{\mathcal{V}_n^c}\prod_{i=1}^n\frac{p(\mathbf{y}_i \mid \boldsymbol{\Sigma})}{p(\mathbf{y}_i \mid \boldsymbol{\Sigma}_0)}\Pi(\mathrm{d}\boldsymbol{\Sigma})\right\}.
\end{aligned}$$

Observe that by Fubini's theorem,

$$\begin{aligned}
&\mathbb{E}_0\left\{(1-\phi_n)\int_{\mathcal{V}_n^c}\prod_{i=1}^n\frac{p(\mathbf{y}_i \mid \boldsymbol{\Sigma})}{p(\mathbf{y}_i \mid \boldsymbol{\Sigma}_0)}\Pi(\mathrm{d}\boldsymbol{\Sigma})\right\}\\
&\le \mathbb{E}_0\left\{(1-\phi_n)\int_{\mathcal{V}_n^c\cap\mathcal{G}_n}\prod_{i=1}^n\frac{p(\mathbf{y}_i \mid \boldsymbol{\Sigma})}{p(\mathbf{y}_i \mid \boldsymbol{\Sigma}_0)}\Pi(\mathrm{d}\boldsymbol{\Sigma})\right\} + \mathbb{E}_0\left\{\int_{\mathcal{G}_n^c}\prod_{i=1}^n\frac{p(\mathbf{y}_i \mid \boldsymbol{\Sigma})}{p(\mathbf{y}_i \mid \boldsymbol{\Sigma}_0)}\Pi(\mathrm{d}\boldsymbol{\Sigma})\right\}\\
&= \int_{\mathcal{V}_n^c\cap\mathcal{G}_n}\mathbb{E}_0\left\{(1-\phi_n)\prod_{i=1}^n\frac{p(\mathbf{y}_i \mid \boldsymbol{\Sigma})}{p(\mathbf{y}_i \mid \boldsymbol{\Sigma}_0)}\right\}\Pi(\mathrm{d}\boldsymbol{\Sigma}) + \int_{\mathcal{G}_n^c}\left\{\mathbb{E}_0\prod_{i=1}^n\frac{p(\mathbf{y}_i \mid \boldsymbol{\Sigma})}{p(\mathbf{y}_i \mid \boldsymbol{\Sigma}_0)}\right\}\Pi(\mathrm{d}\boldsymbol{\Sigma})\\
&\le \int_{\mathcal{V}_n^c\cap\mathcal{G}_n}\mathbb{E}_{\boldsymbol{\Sigma}}(1-\phi_n)\Pi(\mathrm{d}\boldsymbol{\Sigma}) + \Pi(\mathcal{G}_n^c)\\
&\le 4\exp\left[-\left\{\frac{C_6 M}{C_\mu^2\|\boldsymbol{\Sigma}_0\|_2^2}\min\left(\frac{\|\boldsymbol{\Sigma}_0\|_2^2}{8},\frac{1}{32}\right)-4(\beta+2)\right\}s\log p\right]\\
&\quad + 7\exp\left\{-\min\left(\frac{\beta\kappa}{2},\frac{\beta}{2\mathrm{e}}-2\gamma\right)s\log p\right\},
\end{aligned}$$

where the testing type II error probability bound (5.9) and (5.7) are applied to the last inequality. Then by taking $M = M_\infty = \max(M_{\infty 1}, M_{\infty 2})$, where

$$\beta = \max\left[\frac{4}{\kappa}C_{0\lambda}, 2\mathrm{e}\left\{2\gamma + 2C_{0\lambda}\right\}\right],$$

$$
\begin{aligned}
M_{\infty 1} &= \max\left(\frac{32C_\mu^2\|\mathbf{\Sigma}_0\|_2^2}{C_6}, \frac{8C_\mu^2}{C_6}\right)\{4(\beta+2)+2C_{0\lambda}\}, \\
M_{\infty 2} &= \max\left(\frac{4C_\mu^4\|\mathbf{\Sigma}_0\|_2^4}{C_6^2}, \frac{2C_\mu^4}{C_6^2}\right)\{C_{0\lambda}+6(\beta+2)\}^2, ,
\end{aligned}
$$

we obtain the following result:

$$
\begin{aligned}
&\mathbb{E}_0\left[(1-\phi_n)\left\{\frac{N_n(\mathcal{V}_n^c)}{D_n}\right\}\mathbb{1}(\mathcal{A}_n)\right] \\
&\quad\le \frac{4}{C(\sigma_0^2)}\exp\left[-\left\{\frac{C_6 M}{C_\mu^2\|\mathbf{\Sigma}_0\|_2^2}\min\left(\frac{\|\mathbf{\Sigma}_0\|_2^2}{8}, \frac{1}{32}\right) - 4(\beta+2) - C_{0\lambda}\right\}s\log p\right] \\
&\qquad + \frac{7}{C(\sigma_0^2)}\exp\left[-\left\{\min\left(\frac{\beta\kappa}{2}, \frac{\beta}{2\mathrm{e}} - 2\gamma\right) - C_{0\lambda}\right\}s\log p\right] \\
&\quad\le \frac{11}{C(\sigma_0^2)}\exp\left(-C_{0\lambda}s\log p\right).
\end{aligned}
$$

Combining the above results, we finally obtain

$$
\begin{aligned}
\mathbb{E}_0\{\Pi(\mathcal{V}_n^c \mid \mathbf{Y}_n)\} &\le \left\{3 + \frac{11}{C(\sigma_0^2)}\right\}\exp\left(-C_{0\lambda}s\log p\right) \\
&\quad + 2\exp\left\{-\tilde{C}_3\min\left(1, \left\|\mathbf{\Sigma}_0^{-1}\right\|_2^{-2}\right)s\log p\right\} \\
&\le \left\{5 + \frac{11}{C(\sigma_0^2)}\right\}\exp\left\{-\min\left(C_{0\lambda}, \tilde{C}_3, \tilde{C}_3\|\mathbf{\Sigma}_0^{-1}\|_2^{-2}\right)s\log p\right\} \\
&= R_0\exp(-C_0 s\log p)
\end{aligned}
$$

by taking $C_0 = \min\left\{C_{0\lambda}, \tilde{C}_3, \tilde{C}_3\|\mathbf{\Sigma}_0^{-1}\|_2^{-2}\right\}$ and $R_0 = \{5 + 11/C(\sigma_0^2)\}$. Therefore, there exists some constant M_∞, such that for all sufficiently large n, we have

$$
\begin{aligned}
\mathbb{E}_0\left\{\Pi\left(\|\mathbf{\Sigma}-\mathbf{\Sigma}_0\|_\infty > M\epsilon_n \,\middle|\, \mathbf{Y}_n\right)\right\} &\le \mathbb{E}_0\left\{\Pi\left(\|\mathbf{\Sigma}-\mathbf{\Sigma}_0\|_\infty > M_0\epsilon_n \,\middle|\, \mathbf{Y}_n\right)\right\} \\
&\le R_0\mathrm{e}^{-C_0 s\log p}
\end{aligned}
$$

for some absolute constants C_0 and R_0 depending on $\mathbf{\Lambda}_0$ and the hyperparameters only whenever $M \ge M_\infty$. Notice that C_0 and R_0 remain the same with those appearing in Theorem 3.

Step 4: Bounding the two-to-infinity norm loss using the Neumann trick. Let $\mathbf{B}\mathbf{B}^{\mathrm{T}} = \mathbf{U}_{\mathbf{B}}\mathbf{\Lambda}\mathbf{U}_{\mathbf{B}}^{\mathrm{T}}$ be the compact spectral decomposition of $\mathbf{B}\mathbf{B}^{\mathrm{T}}$. Denote

$\mathbf{E} = \mathbf{B}\mathbf{B}^{\mathrm{T}} - \mathbf{U}_0\mathbf{\Lambda}_0\mathbf{U}_0^{\mathrm{T}}$ to be the "error" matrix. Clearly, $(\mathbf{U}_0\mathbf{\Lambda}_0\mathbf{U}_0^{\mathrm{T}} + \mathbf{E})\mathbf{U}_{\mathbf{B}} = (\mathbf{U}_{\mathbf{B}}\mathbf{\Lambda}\mathbf{U}_{\mathbf{B}}^{\mathrm{T}})\mathbf{U}_{\mathbf{B}} = \mathbf{U}_{\mathbf{B}}\mathbf{\Lambda}$ by definition, yielding the matrix Sylvester equation

$$\mathbf{U}_{\mathbf{B}}\mathbf{\Lambda} - \mathbf{E}\mathbf{U}_{\mathbf{B}} = (\mathbf{U}_0\mathbf{\Lambda}_0\mathbf{U}_0^{\mathrm{T}})\mathbf{U}_{\mathbf{B}}.$$

Now consider the events

$$\mathcal{U}_n = \left\{\|\mathbf{\Sigma} - \mathbf{\Sigma}_0\|_2 \leq M_0\sqrt{\frac{s\log p}{n}}\right\}, \quad \mathcal{V}_n = \left\{\|\mathbf{\Sigma} - \mathbf{\Sigma}_0\|_\infty \leq M_\infty r\sqrt{\frac{s\log p}{n}}\right\}.$$

Suppose $\mathbf{\Sigma} \in \mathcal{U}_n \cap \mathcal{V}_n$. By the Weyl's inequality, for sufficiently large n, we have

$$\begin{aligned}
|\sigma^2 - \sigma_0^2| &= |\lambda_{r+1}(\mathbf{\Sigma}) - \lambda_{r+1}(\mathbf{\Sigma}_0)| \leq \max_{k\in[p]}|\lambda_k(\mathbf{\Sigma}) - \lambda_k(\mathbf{\Sigma}_0)| \leq M_0\sqrt{\frac{s\log p}{n}},\\
\lambda_r(\mathbf{\Lambda}) &\geq \lambda_{0r} - |\lambda_{0r} - \lambda_r(\mathbf{\Lambda})| \geq \lambda_{0r} - |(\lambda_{0r} + \sigma_0^2) - \{\lambda_r(\mathbf{\Lambda}) + \sigma^2\}| - |\sigma_0^2 - \sigma^2|\\
&\geq \lambda_{0r} - \max_{k\in[p]}|\lambda_k(\mathbf{\Sigma}) - \lambda_k(\mathbf{\Sigma}_0)| - M_0\sqrt{\frac{s\log p}{n}}\\
&\geq \lambda_{0r} - 2M_0\sqrt{\frac{s\log p}{n}} > \max\left\{\frac{\lambda_{0r}}{2}, 2M_0\sqrt{\frac{s\log p}{n}}\right\},\\
\|\mathbf{E}\|_2 &\leq \|\mathbf{\Sigma} - \mathbf{\Sigma}_0\|_2 + \|(\sigma^2 - \sigma_0^2)\mathbf{I}_p\|_2 \leq 2M_0\sqrt{\frac{s\log p}{n}}.
\end{aligned}$$

Therefore, the spectra of $\mathbf{\Lambda}$ and $\mathbf{E}$ are disjoint, and we can apply the Neumann's trick (see Theorem VII.2.2 in [83]) to expand $\mathbf{U}_{\mathbf{B}}$ in terms of a matrix series:

$$\mathbf{U}_{\mathbf{B}} = \sum_{m=0}^{\infty}\mathbf{E}^m(\mathbf{U}_0\mathbf{\Lambda}_0\mathbf{U}_0^{\mathrm{T}})\mathbf{U}_{\mathbf{B}}\mathbf{\Lambda}^{-(m+1)} \tag{5.10}$$

Now we proceed to bound $\|\mathbf{U}_{\mathbf{B}} - \mathbf{U}_0\mathbf{W}_{\mathbf{U}}\|_{2\to\infty}$ using the techniques developed in [85]. Write

$$\begin{aligned}
\mathbf{U}_{\mathbf{B}} - \mathbf{U}_0\mathbf{W}_{\mathbf{U}} &= (\mathbf{U}_{\mathbf{B}}\mathbf{\Lambda}\mathbf{U}_{\mathbf{B}}^{\mathrm{T}} - \mathbf{U}_0\mathbf{\Lambda}_0\mathbf{U}_0^{\mathrm{T}})\mathbf{U}_{\mathbf{B}}\mathbf{\Lambda}^{-1} + \mathbf{U}_0\mathbf{\Lambda}_0(\mathbf{U}_0^{\mathrm{T}}\mathbf{U}_{\mathbf{B}}\mathbf{\Lambda}^{-1} - \mathbf{\Lambda}_0^{-1}\mathbf{U}_0^{\mathrm{T}}\mathbf{U}_{\mathbf{B}})\\
&\quad + \mathbf{U}_0(\mathbf{U}_0^{\mathrm{T}}\mathbf{U}_{\mathbf{B}} - \mathbf{W}_{\mathbf{U}})\\
&= \mathbf{E}\mathbf{U}_{\mathbf{B}}\mathbf{\Lambda}^{-1} + \mathbf{U}_0\mathbf{\Lambda}_0(\mathbf{U}_0^{\mathrm{T}}\mathbf{U}_{\mathbf{B}}\mathbf{\Lambda}^{-1} - \mathbf{\Lambda}_0^{-1}\mathbf{U}_0^{\mathrm{T}}\mathbf{U}_{\mathbf{B}}) + \mathbf{U}_0(\mathbf{U}_0^{\mathrm{T}}\mathbf{U}_{\mathbf{B}} - \mathbf{W}_{\mathbf{U}}).
\end{aligned}$$

By the CS decomposition and the sine-theta theorem, we see that the third term can be bounded:

$$\|\mathbf{U}_0(\mathbf{U}_0^{\mathrm{T}}\mathbf{U}_{\mathbf{B}} - \mathbf{W}_{\mathbf{U}})\|_{2\to\infty} \leq \|\mathbf{U}_0\|_{2\to\infty}\|\mathbf{U}_{\mathbf{B}}\mathbf{U}_{\mathbf{B}}^{\mathrm{T}} - \mathbf{U}_0\mathbf{U}_0^{\mathrm{T}}\|_2^2 \leq \frac{4M_0^2C_\mu}{\lambda_{0r}^2}\left(\frac{\sqrt{rs}\log p}{n}\right).$$

Now we consider the second term. Denote $\mathbf{R} = \mathbf{U}_0^{\mathrm{T}}\mathbf{U}_{\mathbf{B}}\mathbf{\Lambda}^{-1} - \mathbf{\Lambda}_0^{-1}\mathbf{U}_0^{\mathrm{T}}\mathbf{U}_{\mathbf{B}}$. Then the (i,j)-th element of $\mathbf{R}$ can be represented as

$$r_{k\ell} = (\mathbf{U}_0)_{*k}^{\mathrm{T}}(\mathbf{U}_{\mathbf{B}})_{*\ell}\left\{\frac{1}{\lambda_\ell(\mathbf{\Lambda})} - \frac{1}{\lambda_{0k}}\right\} = \frac{1}{\lambda_\ell(\mathbf{\Lambda})\lambda_{0k}}\{\lambda_{0k} - \lambda_\ell(\mathbf{\Lambda})\}(\mathbf{U}_0)_{*k}^{\mathrm{T}}(\mathbf{U}_{\mathbf{B}})_{*\ell}.$$

Therefore, by defining $\mathbf{H}_1 \in \mathbb{R}^{r\times r}$ by $(h_1)_{k\ell} = 1/\{\lambda_\ell(\mathbf{\Lambda})\lambda_{0k}\}$, we have

$$\begin{aligned}\|\mathbf{R}\|_2 &= \|\mathbf{H}_1 \circ (\mathbf{U}_0^{\mathrm{T}}\mathbf{U}_{\mathbf{B}}\mathbf{\Lambda} - \mathbf{\Lambda}_0\mathbf{U}_0^{\mathrm{T}}\mathbf{U}_{\mathbf{B}})\|_2 \le r\|\mathbf{H}_1\|_{\max}\|\mathbf{U}_0^{\mathrm{T}}\mathbf{E}\mathbf{U}_{\mathbf{B}}\|_2 \\ &\le r\|\mathbf{H}_1\|_{\max}2M_0\sqrt{\frac{s\log p}{n}},\end{aligned} \tag{5.11}$$

where $\circ$ represents the Hadamard matrix product (element-wise product), and $\|\cdot\|_{\max}$ is the maximum of the absolute values of the entries of a matrix. Furthermore, using the Weyl's inequality, we have

$$\|\mathbf{H}_1\|_{\max} \le \frac{1}{\lambda_r(\mathbf{\Lambda})\lambda_{0r}} \le \frac{2}{\lambda_{0r}^2}$$

for sufficiently large n, since $\|\mathbf{\Lambda}^{-1}\|_2 = 1/\lambda_r(\mathbf{\Lambda}) \le 2/\lambda_{0r}$ for sufficiently large n. Hence, the second term can be bounded:

$$\|\mathbf{U}_0\mathbf{\Lambda}_0(\mathbf{U}_0^{\mathrm{T}}\mathbf{U}_{\mathbf{B}}\mathbf{\Lambda}^{-1} - \mathbf{\Lambda}_0^{-1}\mathbf{U}_0^{\mathrm{T}}\mathbf{U}_{\mathbf{B}})\|_{2\to\infty} = \|\mathbf{U}_0\|_{2\to\infty}\|\mathbf{\Lambda}_0\|_2\|\mathbf{R}\|_2 \le \frac{4M_0C_\mu\lambda_{01}}{\lambda_{0r}^2}\sqrt{\frac{r^3\log p}{n}}.$$

Now we focus on the first term. By the Neumann matrix series (5.10), we have

$$\begin{aligned}\|\mathbf{E}\mathbf{U}_{\mathbf{B}}\mathbf{\Lambda}^{-1}\|_{2\to\infty} &= \left\|\sum_{m=1}^{\infty}\mathbf{E}^m(\mathbf{U}_0\mathbf{\Lambda}_0\mathbf{U}_0^{\mathrm{T}})\mathbf{U}_{\mathbf{B}}\mathbf{\Lambda}^{-(m+1)}\right\|_{2\to\infty} \\ &\le \|\mathbf{E}\mathbf{U}_0\|_{2\to\infty}\|\mathbf{\Lambda}_0\|_2\|\|\mathbf{\Lambda}^{-1}\|_2^2 + \sum_{m=2}^{\infty}\|\mathbf{E}\|_2^m\|\mathbf{\Lambda}_0\|_2\|\mathbf{\Lambda}^{-1}\|_2^{(m+1)} \\ &\le \|\mathbf{E}\mathbf{U}_0\|_{2\to\infty}\left\{\frac{\lambda_{01}}{\lambda_r(\mathbf{\Lambda})^2}\right\} + \left\{\frac{\lambda_{01}}{\lambda_r(\mathbf{\Lambda})}\right\}\frac{\|\mathbf{E}\|_2^2\|\mathbf{\Lambda}^{-1}\|_2^2}{1 - \|\mathbf{E}\|_2\|\mathbf{\Lambda}^{-1}\|_2} \\ &\le 4\|\mathbf{E}\|_\infty\|\mathbf{U}_0\|_{2\to\infty}\frac{\lambda_{01}}{\lambda_{0r}^2} + \frac{8\lambda_{01}}{\lambda_{0r}^3}\|\mathbf{E}\|_2^2 \\ &\le \frac{4M_\infty C_\mu\lambda_{01}}{\lambda_{0r}^2}\sqrt{\frac{r^3\log p}{n}} + \frac{8M_0^2\lambda_{01}}{\lambda_{0r}^3}\frac{s\log p}{n}\end{aligned}$$

for sufficiently large n. In other words, there exists some constant $M_{2\to\infty}$ depending on M_0, M_∞, $\mathbf{\Lambda}_0$, and hyperparameters, such that

$$\|\mathbf{U}_{\mathbf{B}} - \mathbf{U}_0\mathbf{W}_{\mathbf{U}}\|_{2\to\infty} \le M_{2\to\infty}\max\left(\sqrt{\frac{r^3\log p}{n}}, \frac{s\log p}{n}\right)$$

for sufficiently large n whenever $\mathbf{\Sigma} \in \mathcal{U}_n \cap \mathcal{V}_n$. Therefore,

$$\begin{aligned}
&\mathbb{E}_0 \left[\Pi \left\{ \|\mathbf{U}_{\mathbf{B}} - \mathbf{U}_0 \mathbf{W}_{\mathbf{U}}\|_{2\to\infty} > M \max\left(\sqrt{\frac{r^3 \log p}{n}}, \frac{s \log p}{n} \right) \right\} \right] \\
&\quad \le \mathbb{E}_0 \left[\Pi \left\{ \|\mathbf{U}_{\mathbf{B}} - \mathbf{U}_0 \mathbf{W}_{\mathbf{U}}\|_{2\to\infty} > M_{2\to\infty} \max\left(\sqrt{\frac{r^3 \log p}{n}}, \frac{s \log p}{n} \right) \right\} \right] \\
&\quad \le \mathbb{E}_0 \left\{ \Pi(\mathcal{U}_n^c \mid \mathbf{Y}_n) + \Pi(\mathcal{V}_n^c \mid \mathbf{Y}_n) \right\} \le 2R_0 \mathrm{e}^{-C_0 s \log p},
\end{aligned}$$

for sufficiently large n when $M \ge M_{2\to\infty}$, completing the proof. □

Proof of Theorem 5. For any random matrix $\mathbf{X} \in \mathbb{R}^{p\times p}$, we have

$$\|\mathbf{E}(\mathbf{X})\|_2^2 = \max_{\|\mathbf{u}\|_2 = 1} \{\mathbb{E}(\mathbf{X}\mathbf{u})\}^{\mathrm{T}} \{\mathbb{E}(\mathbf{X}\mathbf{u})\} \le \mathbb{E}\|\mathbf{X}\|_2^2$$

by the Jensen's inequality. Now take $\mathbf{X} = \mathbf{U}_{\mathbf{B}}\mathbf{U}_{\mathbf{B}}^{\mathrm{T}} - \mathbf{U}_0\mathbf{U}_0^{\mathrm{T}}$. Denote the event $\mathcal{U}_n = \{\|\mathbf{U}_{\mathbf{B}}\mathbf{U}_{\mathbf{B}}^{\mathrm{T}} - \mathbf{U}_0\mathbf{U}_0^{\mathrm{T}}\|_2 \le M_0\epsilon_n\}$. Invoking the posterior contraction (2.7), we have

$$\begin{aligned}
\mathbb{E}_0 \left(\left\| \widehat{\mathbf{\Omega}} - \mathbf{U}_0\mathbf{U}_0^{\mathrm{T}} \right\|_2^2 \right) &= \mathbb{E}_0 \left\{ \left\| \int \left(\mathbf{U}_{\mathbf{B}}\mathbf{U}_{\mathbf{B}}^{\mathrm{T}} - \mathbf{U}_0\mathbf{U}_0^{\mathrm{T}} \right) \Pi(\mathrm{d}\mathbf{B} \mid \mathbf{Y}_n) \right\|_2^2 \right\} \\
&\le \mathbb{E}_0 \left\{ \int_{\mathcal{U}_n} \left\| \left(\mathbf{U}_{\mathbf{B}}\mathbf{U}_{\mathbf{B}}^{\mathrm{T}} - \mathbf{U}_0\mathbf{U}_0^{\mathrm{T}} \right) \right\|_2^2 \Pi(\mathrm{d}\mathbf{B} \mid \mathbf{Y}_n) \right\} \\
&\quad + \mathbb{E}_0 \left\{ \int_{\mathcal{U}_n^c} \left\| \left(\mathbf{U}_{\mathbf{B}}\mathbf{U}_{\mathbf{B}}^{\mathrm{T}} - \mathbf{U}_0\mathbf{U}_0^{\mathrm{T}} \right) \right\|_2^2 \Pi(\mathrm{d}\mathbf{B} \mid \mathbf{Y}_n) \right\} \\
&\le M_0^2 \epsilon_n^2 + \left(\sup_{\mathbf{U} \in \mathbb{O}(p,r)} \|\mathbf{U}\mathbf{U}^{\mathrm{T}} - \mathbf{U}_0\mathbf{U}_0^{\mathrm{T}}\|_2^2 \right) \mathbb{E}_0 \left\{ \Pi(\mathcal{U}_n^c \mid \mathbf{Y}_n) \right\} \\
&\le \frac{4M_0^2}{\lambda_{0r}^2} \epsilon_n^2 + 4R_0 \exp(-C_0 s \log p).
\end{aligned}$$

Since for sufficiently large n, we have

$$\epsilon_n^2 = \frac{s \log p}{n} = \exp\left(\log s + \log\log p - \log n \right) \ge \exp(-C_0 s \log p),$$

we obtain

$$\mathbb{E}_0 \left(\left\| \widehat{\mathbf{\Omega}} - \mathbf{U}_0\mathbf{U}_0^{\mathrm{T}} \right\|_2 \right) \le \left\{ \mathbb{E}_0 \left(\left\| \widehat{\mathbf{\Omega}} - \mathbf{U}_0\mathbf{U}_0^{\mathrm{T}} \right\|_2^2 \right) \right\}^{1/2} \le \epsilon_n \left(\frac{2M_0}{\lambda_{0r}} + 2\sqrt{R_0} \right).$$

Since the columns of $\widehat{\mathbf{U}}$ are the leading r-eigenvectors of $\widehat{\mathbf{\Omega}}$ corresponding to $\lambda_1(\widehat{\mathbf{\Omega}})$, ..., $\lambda_r(\widehat{\mathbf{\Omega}})$, *i.e.*, $\widehat{\mathbf{\Omega}}\widehat{\mathbf{U}}_{*k} = \lambda_k(\widehat{\mathbf{\Omega}})\widehat{\mathbf{U}}_{*k}$, then applying the sine-theta theorem (Theorem

21) yields

$$\mathbb{E}_0\left(\|\widehat{\mathbf{U}}\widehat{\mathbf{U}}^{\mathrm{T}} - \mathbf{U}_0\mathbf{U}_0^{\mathrm{T}}\|_2\right) \leq \left(\frac{4M_0}{\lambda_{0r}} + 4\sqrt{R_0}\right)\epsilon_n.$$

□

5.1.3 Proofs of results in Section 2.4.3

Proof of Lemma 5. To prove Lemma 5, we need the following auxiliary matrix inequality:

Lemma 9 (Supplement Lemma 1.3 in [41]) *Let $\mathbf{\Sigma}, \mathbf{\Sigma}_0$ be $p \times p$ positive definite matrices and $\eta \in (0,1)$. If $\|\mathbf{\Sigma} - \mathbf{\Sigma}_0\|_{\mathrm{F}} \leq \eta$ and $\eta < 2\lambda_r(\mathbf{\Sigma}_0)$, then*

$$\log\det\left(\mathbf{\Sigma}_0\mathbf{\Sigma}^{-1}\right) - \mathrm{tr}\left(\mathbf{\Sigma}_0\mathbf{\Sigma}^{-1} - \mathbf{I}\right) \geq -C_3\frac{\eta^2\log\rho}{\lambda_r(\mathbf{\Sigma}_0)}$$

for some absolute constant $C_3 > 0$, where $\rho = 2\lambda_1(\mathbf{\Sigma}_0)/\lambda_r(\mathbf{\Sigma}_0)$.

Denote $\Pi\{\cdot \mid \mathcal{K}_n(\eta)\} = \Pi\{\cdot \cap \mathcal{K}_n)/\Pi_n(\mathcal{K}_n(\eta)\}$ to be the re-normalized restriction of Π on $\mathcal{K}_n(\eta)$. Define random variable

$$\begin{aligned}
w_{ni} &= \int \log\frac{p(\mathbf{y}_i \mid \mathbf{\Sigma})}{p(\mathbf{y}_i \mid \mathbf{\Sigma}_0)}\Pi\{\mathrm{d}\mathbf{\Sigma} \mid \mathcal{K}_n(\eta)\} \\
&= \int\left\{\frac{1}{2}\log\det(\mathbf{\Sigma}_0\mathbf{\Sigma}^{-1})\right\}\Pi\{\mathrm{d}\mathbf{\Sigma} \mid \mathcal{K}_n(\eta)\} + \frac{1}{2}\mathbf{y}_i^{\mathrm{T}}\left[\int\left(\mathbf{\Sigma}_0^{-1} - \mathbf{\Sigma}^{-1}\right)\Pi\{\mathrm{d}\mathbf{\Sigma} \mid \mathcal{K}_n(\eta)\}\right]\mathbf{y}_i.
\end{aligned}$$

Invoking Fubini's theorem and Lemma 9, we derive

$$\begin{aligned}
\mathbb{E}_0(w_{ni}) &= \int\left\{\frac{1}{2}\log\det(\mathbf{\Sigma}_0\mathbf{\Sigma}^{-1})\right\}\Pi\{\mathrm{d}\mathbf{\Sigma} \mid \mathcal{K}_n(\eta)\} \\
&\quad + \frac{1}{2}\int\mathbb{E}_0\left\{\mathbf{y}_i^{\mathrm{T}}\left(\mathbf{\Sigma}_0^{-1} - \mathbf{\Sigma}^{-1}\right)\mathbf{y}_i\right\}\Pi\{\mathrm{d}\mathbf{\Sigma} \mid \mathcal{K}_n(\eta)\} \\
&= \frac{1}{2}\int\left\{\log\det\left(\mathbf{\Sigma}_0\mathbf{\Sigma}^{-1}\right) + \mathrm{tr}\left(\mathbf{I} - \mathbf{\Sigma}_0\mathbf{\Sigma}^{-1}\right)\right\}\Pi\{\mathrm{d}\mathbf{\Sigma} \mid \mathcal{K}_n(\eta)\} \\
&\geq -\frac{C_3\log\rho}{2(\lambda_{0r} + \sigma_0^2)}\eta^2.
\end{aligned}$$

Hence by Jensen's inequality,

$$\log D_n - \log\Pi\{\mathbf{\Sigma} \in \mathcal{K}_n(\eta)\} \geq \log\left[\int_{\mathcal{K}_n(\eta)}\exp\left\{\ell_n(\mathbf{\Sigma}) - \ell_n(\mathbf{\Sigma}_0)\right\}\frac{\Pi(\mathrm{d}\mathbf{\Sigma})}{\Pi\{\mathcal{K}_n(\eta)\}}\right]$$

$$
\begin{aligned}
&= \log\left[\int \exp\left\{\ell_n(\mathbf{\Sigma}) - \ell_n(\mathbf{\Sigma}_0)\right\}\Pi\{\mathrm{d}\mathbf{\Sigma} \mid \mathcal{K}_n(\eta)\}\right]\\
&\geq \int \{\ell_n(\mathbf{\Sigma}) - \ell_n(\mathbf{\Sigma}_0)\}\Pi\{\mathrm{d}\mathbf{\Sigma} \mid \mathcal{K}_n(\eta)\}\\
&= n\mathbb{E}_0(w_{ni}) + \sum_{i=1}^n \{w_{ni} - \mathbb{E}_0(w_{ni})\}\\
&\geq -\frac{C_3\log\rho}{2(\lambda_{0r}+\sigma_0^2)}n\eta^2 + \sum_{i=1}^n\{w_{ni} - \mathbb{E}_0(w_{ni})\}.
\end{aligned}
$$

Now let $\mathcal{A}_n = \{|\sum_{i=1}^n\{w_{ni} - \mathbb{E}_0(w_{ni})\}| \leq n\eta^2\}$. Clearly,

$$
\begin{aligned}
\mathcal{A}_n &\subset \left\{\log D_n - \log\Pi\{\mathbf{\Sigma}\in\mathcal{K}_n(\eta)\} \geq -\left\{\frac{C_3\log\rho}{2(\lambda_{0r}+\sigma_0^2)} + 1\right\}n\eta^2\right\}\\
&= \left\{D_n \geq \Pi\{\mathbf{\Sigma}\in\mathcal{K}_n(\eta)\}\exp\left[-\left\{\frac{C_3\log\rho}{2(\lambda_{0r}+\sigma_0^2)} + 1\right\}n\eta^2\right]\right\}.
\end{aligned}
$$

We now analyze the probabilistic bound of $\mathcal{A}_n^c$. Recall $\mathbf{\Sigma}_0 = \mathbf{U}_0\mathbf{\Lambda}_0\mathbf{U}_0^{\mathrm{T}} + \sigma_0^2\mathbf{I}_p$. Let $\mathbf{U}_{0\perp}$ to be the orthonormal $(p-r)$-frame in $\mathbb{R}^p$ such that $[\mathbf{U}_0, \mathbf{U}_{0\perp}] \in \mathbb{O}(p)$, and denote

$$
\mathbf{\Sigma}_0^{1/2} = [\mathbf{U}_0, \mathbf{U}_{0\perp}]\mathrm{diag}\{\lambda_1(\mathbf{\Sigma}_0)^{1/2}, \ldots, \lambda_p(\mathbf{\Sigma}_0)^{1/2}\}[\mathbf{U}_0, \mathbf{U}_{0\perp}]^{\mathrm{T}}.
$$

Clearly, $\mathbf{\Sigma}_0 = (\mathbf{\Sigma}_0^{1/2})^2$, and by denoting $\mathbf{v}_i = \mathbf{\Sigma}_0^{-1/2}\mathbf{y}_i$, we have $\mathbf{v}_i \sim \mathrm{N}_p(\mathbf{0}_p, \mathbf{I}_p)$ under $\mathbb{P}_0$. Re-writing $w_{ni} - \mathbb{E}_0(w_{ni})$ in terms of $\mathbf{v}_i$, we have

$$
w_{ni} - \mathbb{E}_0(w_{ni}) = \mathbf{v}_i^{\mathrm{T}}\mathbf{\Omega}\mathbf{v}_i - \mathbb{E}_0(\mathbf{v}_i\mathbf{\Omega}\mathbf{v}_i),
$$

where

$$
\mathbf{\Omega} = \frac{1}{2}\int\left(\mathbf{I}_p - \mathbf{\Sigma}_0^{1/2}\mathbf{\Sigma}^{-1}\mathbf{\Sigma}_0^{1/2}\right)\Pi\{\mathrm{d}\mathbf{\Sigma} \mid \mathcal{K}_n(\eta)\}.
$$

Let $\mathbf{\Omega} = \mathbf{U}_{\mathbf{\Omega}}\mathbf{D}_{\mathbf{\Omega}}\mathbf{U}_{\mathbf{\Omega}}^{\mathrm{T}}$ be the spectral decomposition of $\mathbf{\Omega}$, and let $\mathbf{x}_i = \mathbf{U}_{\mathbf{\Omega}}^{\mathrm{T}}\mathbf{v}_i$. Then we proceed to bound

$$
\begin{aligned}
\mathbb{P}_0(\mathcal{A}_n^c) &\leq \mathbb{P}_0\left(\left|\sum_{i=1}^n\{w_{ni} - \mathbb{E}_0(w_{ni})\}\right| \geq n\eta^2\right)\\
&= \mathbb{P}_0\left(\left|\sum_{i=1}^n\left\{\mathbf{x}_i^{\mathrm{T}}\mathbf{D}_{\mathbf{\Omega}}\mathbf{x}_i - \mathbb{E}_0\left(\mathbf{x}_i^{\mathrm{T}}\mathbf{D}_{\mathbf{\Omega}}\mathbf{x}_i\right)\right\}\right| \geq n\eta^2\right)\\
&= \mathbb{P}_0\left(\left|\sum_{i=1}^n\sum_{j=1}^p\lambda_j(\mathbf{\Omega})\left\{x_{ij}^2 - \mathbb{E}_0(x_{ij}^2)\right\}\right| \geq n\eta^2\right)\\
&\leq 2\exp\left[-C_3'\min\left\{\frac{n^2\eta^4}{n\sum_{j=1}^p\lambda_j(\mathbf{\Omega})^2}, \frac{n\eta^2}{\max_{j\in[p]}\lambda_j(\mathbf{\Omega})}\right\}\right]
\end{aligned}
$$

for some absolute constant $C_3' > 0$, where the large deviation inequality for sub-exponential random variables is applied again in the last inequality. Observe that over $\mathcal{K}_n(\eta)$ for $\eta \leq \sigma_0^2/2$,

$$\begin{aligned}
\|\mathbf{\Sigma}^{-1}\|_2 &\leq \|\mathbf{\Sigma}^{-1} - \mathbf{\Sigma}_0^{-1}\|_2 + \|\mathbf{\Sigma}_0^{-1}\|_2 = \|\mathbf{\Sigma}^{-1}(\mathbf{\Sigma} - \mathbf{\Sigma}_0)\mathbf{\Sigma}_0^{-1}\|_2 + \|\mathbf{\Sigma}_0^{-1}\|_2 \\
&\leq \|\mathbf{\Sigma}^{-1}\|_2\|\mathbf{\Sigma} - \mathbf{\Sigma}_0\|_{\mathrm{F}}\|\mathbf{\Sigma}_0^{-1}\|_2 + \|\mathbf{\Sigma}_0^{-1}\|_2 \leq \frac{\eta}{\sigma_0^2}\|\mathbf{\Sigma}^{-1}\|_2 + \|\mathbf{\Sigma}_0^{-1}\|_2,
\end{aligned}$$

implying that $\|\mathbf{\Sigma}^{-1}\|_2 \leq 2\|\mathbf{\Sigma}_0^{-1}\|_2$. Also observe that

$$\begin{aligned}
\sum_{j=1}^{p} \lambda_j(\mathbf{\Omega})^2 = \|\mathbf{\Omega}\|_{\mathrm{F}}^2 &\leq \frac{1}{4}\int \left\|\mathbf{I}_p - \mathbf{\Sigma}_0^{1/2}\mathbf{\Sigma}^{-1}\mathbf{\Sigma}_0^{1/2}\right\|_{\mathrm{F}}^2 \Pi\{\mathrm{d}\mathbf{\Sigma} \mid \mathcal{K}_n(\eta)\} \\
&= \frac{1}{4}\int \left\|\mathbf{I}_p - \mathbf{\Sigma}^{-1}\mathbf{\Sigma}_0\right\|_{\mathrm{F}}^2 \Pi\{\mathrm{d}\mathbf{\Sigma} \mid \mathcal{K}_n(\eta)\} \\
&\leq \frac{1}{4}\int \|\mathbf{\Sigma}^{-1}\|_2^2\|\mathbf{\Sigma} - \mathbf{\Sigma}_0\|_{\mathrm{F}}^2\Pi\{\mathrm{d}\mathbf{\Sigma} \mid \mathcal{K}_n(\eta)\} \\
&\leq \|\mathbf{\Sigma}_0^{-1}\|_2^2 \int \|\mathbf{\Sigma} - \mathbf{\Sigma}_0\|_{\mathrm{F}}^2\Pi\{\mathrm{d}\mathbf{\Sigma} \mid \mathcal{K}_n(\eta)\} \\
&\leq \|\mathbf{\Sigma}_0^{-1}\|_2^2\eta^2.
\end{aligned}$$

We finally obtain

$$\mathbb{P}_0(\mathcal{A}_n^c) \leq 2\exp\left\{-\tilde{C}_3 \min\left(\frac{n\eta^2}{\|\mathbf{\Sigma}_0^{-1}\|_2^2}, n\eta^2\right)\right\}$$

for some absolute constant $\tilde{C}_3 > 0$. □

Proof of Lemma 6. To proof Lemma 6, we need the following oracle testing lemma from [40]:

Lemma 10 ([40]) *Let $\mathbf{y}_i \sim \mathrm{N}_d(\mathbf{0}_d, \mathbf{\Sigma}$, where $\mathbf{\Sigma} \in \mathbb{R}^{d\times d}$. Then for any $M > 0$, there exists a test function ϕ_n such that*

$$\begin{aligned}
\mathbb{E}_{\mathbf{\Sigma}^{(1)}}(\phi_n) &\leq \exp\left(C_4 d - \frac{C_4 M^2}{4\|\mathbf{\Sigma}^{(1)}\|_2^2}n\epsilon^2\right) + 2\exp\left(C_4 d - C_4\sqrt{M}n\right), \\
\sup_{\{\mathbf{\Sigma}^{(2)}:\|\mathbf{\Sigma}^{(2)}-\mathbf{\Sigma}^{(1)}\|_2 > M\epsilon\}} \mathbb{E}_{\mathbf{\Sigma}^{(2)}}(1-\phi_n) &\leq \exp\left[C_4 d - \frac{C_4 M n\epsilon^2}{4}\max\left\{1, \frac{M}{(\sqrt{M}+2)^2\|\mathbf{\Sigma}^{(1)}\|_2^2}\right\}\right].
\end{aligned}$$

with some absolute constant $C_4 > 0$.

Let $S_0 = \mathrm{supp}(\mathbf{U}_0)$ and $S(\delta) = \mathrm{supp}_\delta(\mathbf{B})$. Then there exists some permutation matrix $\mathbf{P}$ such that

$$\mathbf{B} = \mathbf{P}\begin{bmatrix}\mathbf{B}_\delta \\ \mathbf{A}_\delta\end{bmatrix} \quad \text{and} \quad \mathbf{U}_0 = \mathbf{P}\begin{bmatrix}\mathbf{U}_{0\delta} \\ \mathbf{0}\end{bmatrix},$$

where $\mathbf{B}_\delta$ and $\mathbf{U}_{0\delta}$ are $|S(\delta)\cup S_0| \times r$ matrices. Hence for $\mathbf{\Sigma} \in \mathcal{F}(\delta, \tau, t)$, it holds that

$$\begin{aligned}
\|\mathbf{\Sigma} - \mathbf{\Sigma}_0\|_2 &= \left\|\mathbf{P}\begin{bmatrix}\mathbf{B}_\delta\mathbf{B}_\delta^{\mathrm{T}} + \sigma^2\mathbf{I} - \mathbf{U}_{0\delta}\mathbf{\Lambda}_0\mathbf{U}_{0\delta}^{\mathrm{T}} - \sigma_0^2\mathbf{I} & \mathbf{B}_\delta\mathbf{A}_\delta^{\mathrm{T}} \\ \mathbf{A}_\delta\mathbf{B}_\delta^{\mathrm{T}} & \mathbf{A}_\delta\mathbf{A}_\delta^{\mathrm{T}} + (\sigma^2 - \sigma_0^2)\mathbf{I}_d\end{bmatrix}\mathbf{P}^{\mathrm{T}}\right\|_2 \\
&\le \left\|\begin{bmatrix}\mathbf{B}_\delta\mathbf{B}_\delta^{\mathrm{T}} + \sigma^2\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \sigma^2\end{bmatrix} - \begin{bmatrix}\mathbf{U}_{0\delta}\mathbf{\Lambda}_0\mathbf{U}_{0\delta}^{\mathrm{T}} + \sigma_0^2\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \sigma_0^2\end{bmatrix}\right\|_2 + \left\|\begin{bmatrix}\mathbf{0} & \mathbf{B}_\delta\mathbf{A}_\delta^{\mathrm{T}} \\ \mathbf{A}_\delta\mathbf{B}_\delta^{\mathrm{T}} & \mathbf{A}_\delta\mathbf{A}_\delta^{\mathrm{T}}\end{bmatrix}\right\|_{\mathrm{F}} \\
&\le \left\|\mathbf{\Sigma}_{S(\delta)} - \mathbf{\Sigma}_{S(\delta)}^{(0)}\right\|_2 + (\|\mathbf{A}_\delta\|_2^2 + 2\|\mathbf{B}_\delta\|_2^2)^{1/2}\|\mathbf{A}_\delta\|_{\mathrm{F}} \\
&\le \left\|\mathbf{\Sigma}_{S(\delta)} - \mathbf{\Sigma}_{S(\delta)}^{(0)}\right\|_2 + (\sqrt{p}\delta + 2t)\sqrt{p}\delta,
\end{aligned}$$

where

$$\mathbf{\Sigma}_{S(\delta)} = \begin{bmatrix}\mathbf{B}_\delta\mathbf{B}_\delta^{\mathrm{T}} + \sigma^2\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \sigma^2\end{bmatrix} \quad \text{and} \quad \mathbf{\Sigma}_{S(\delta)}^{(0)} = \begin{bmatrix}\mathbf{U}_{0\delta}\mathbf{\Lambda}_0\mathbf{U}_{0\delta}^{\mathrm{T}} + \sigma_0^2\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \sigma_0^2\end{bmatrix}.$$

By taking $M \ge 2M_1$, we obtain

$$\begin{aligned}
&\left\{\mathbf{\Sigma} = \mathbf{B}\mathbf{B}^{\mathrm{T}} + \sigma^2\mathbf{I} : \|\mathbf{\Sigma} - \mathbf{\Sigma}_0\| > M\epsilon_n, \mathbf{B} \in \mathcal{F}(\delta_n, \tau_n, t_n)\right\} \\
&\quad \subset \left\{\mathbf{\Sigma} : \left\|\mathbf{\Sigma}_{S(\delta)} - \mathbf{\Sigma}_{S(\delta)}^{(0)}\right\|_2 > \frac{M}{2}\epsilon_n : \mathbf{B} \in \mathcal{F}(\delta_n, \tau_n, t_n)\right\} \\
&\quad \subset \bigcup_{S(\delta_n)\subset[p]:|S(\delta)|\le\tau_n} \left\{\mathbf{\Sigma} : \left\|\mathbf{\Sigma}_{S(\delta)} - \mathbf{\Sigma}_{S(\delta)}^{(0)}\right\|_2 > \frac{M}{2}\epsilon_n\right\}.
\end{aligned}$$

Since both $\mathbf{\Sigma}_{S(\delta_n)}$ and $\mathbf{\Sigma}_{S(\delta_n)}^{(0)}$ are $(|S(\delta_n)\cup S_0|+1)\times(|S(\delta_n)\cup S_0|+1)$ square matrices, and

$$|S(\delta_n)\cup S_0| + 1 \le |S(\delta_n)| + S_0 + 1 \le \tau_n + 2s_n,$$

then for each $S(\delta_n) \subset [p]$ with $|S(\delta_n)| \le \tau_n$, and for each $M \ge \max\{M_1/2, (128\|\mathbf{\Sigma}_0\|_2^4)^{1/3}\}$, we invoke Lemma 10 to construct a test $\phi_{S(\delta_n)}$ depending on the index set $S(\delta_n)$, such that the type I error probability satisfies

$$\begin{aligned}
\mathbb{E}_{\mathbf{\Sigma}_{S(\delta_n)}^{(0)}}\left(\phi_{S(\delta_n)}\right) &\le \exp\left\{C_4(\tau_n + 2s_n) - \frac{C_4M^2n\epsilon_n^2}{16\|\mathbf{\Sigma}_{S(\delta_n)}^{(0)}\|_2^2}\right\} \\
&\quad + 2\exp\left\{C_4(\tau_n + 2s_n) - C_4\sqrt{\frac{M}{2}}n\right\}
\end{aligned}$$

$$
\begin{aligned}
&\leq 3\exp\left\{C_4(\tau_n + 2s_n) - C_4\min\left(\frac{M^2}{16\|\mathbf{\Sigma}_0\|_2^2}, \sqrt{\frac{M}{2}}\right) n\epsilon_n^2\right\}\\
&\leq 3\exp\left\{C_4(\tau_n + 2s_n) - C_4\sqrt{\frac{M}{2}} n\epsilon_n^2\right\},
\end{aligned}
$$

and for all $\mathbf{\Sigma}_{S(\delta_n)} \in \{\|\mathbf{\Sigma}_{S(\delta_n)} - \mathbf{\Sigma}_{S(\delta_n)}^{(0)}\|_2 > M\epsilon_n/2\}$, the type II error probability satisfies

$$
\begin{aligned}
\mathbb{E}_{\mathbf{\Sigma}_{S(\delta_n)}^{(1)}}\left(1 - \phi_{S(\delta_n)}\right) &\leq \exp\left[C_4(\tau_n + 2s_n) - \frac{C_4 M n\epsilon_n^2}{8}\max\left\{1, \frac{M}{(\sqrt{M}+2)^2\|\mathbf{\Sigma}_{S(\delta_n)}^{(0)}\|_2^2}\right\}\right]\\
&\leq \exp\left\{C_4(\tau_n + 2s_n) - \frac{C_4 M n\epsilon_n^2}{8}\right\}.
\end{aligned}
$$

Notice that for each index set $S(\delta_n)$, the test function $\phi_{S(\delta_n)}$ is only a function of $\mathbf{Y}_n$ through the coordinates $[A_{ij} : i \in [n], j \in S(\delta_n) \cup S_0]$. Hence, $\mathbb{E}_{\mathbf{\Sigma}_{S(\delta_n)}^{(0)}}(\phi_{S(\delta_n)}) = \mathbb{E}_0(\phi_{S(\delta_n)})$, and for any $p \times p$ covariance matrix $\mathbf{\Sigma}$ with $\|\mathbf{\Sigma}_{S(\delta_n)} - \mathbf{\Sigma}_{S(\delta_n)}^{(0)}\|_2 > M\epsilon_n/2$, it holds that $\mathbb{E}_{\mathbf{\Sigma}_{S(\delta_n)}}(1 - \phi_{S(\delta_n)}) = \mathbb{E}_{\mathbf{\Sigma}}(1 - \phi_{S(\delta_n)})$. Therefore, by aggregating the test functions

$$
\phi_n = \max_{S(\delta_n)\subset[p]:|S(\delta_n)|\leq\tau_n} \phi_{S(\delta_n)},
$$

we obtain

$$
\begin{aligned}
\mathbb{E}_0(\phi_n) &\leq \sum_{S(\delta_n)\subset[p]:|S(\delta_n)|\leq\tau_n} \mathbb{E}_{\mathbf{\Sigma}_{S(\delta_n)}^{(0)}}(\phi_{S(\delta_n)})\\
&\leq 3\sum_{q=0}^{\lfloor\tau_n\rfloor} \frac{p!}{q!(p-q)!}\exp\left\{C_4(\tau_n + 2s_n) - C_4\sqrt{\frac{M}{2}} n\epsilon_n^2\right\}\\
&\leq 3(\tau_n + 1)\exp(\tau_n\log p)\exp\left\{C_4(\tau_n + 2s_n) - C_4\sqrt{\frac{M}{2}} n\epsilon_n^2\right\}\\
&\leq 3\exp\left\{\tau_n + \tau_n\log p + C_4(\tau_n + 2s_n) - C_4\sqrt{\frac{M}{2}} n\epsilon_n^2\right\}\\
&\leq 3\exp\left\{(2 + C_4)(\tau_n\log p + 2s_n) - C_4\sqrt{\frac{M}{2}} n\epsilon_n^2\right\},
\end{aligned}
$$

and

$$
\sup_{\mathbf{\Sigma}\in H_1}\mathbb{E}_{\mathbf{\Sigma}}(1 - \phi_n) \leq \sup_{S(\delta_n)\subset[p]:|S(\delta_n)|\leq\tau_n}\ \sup_{\left\{\mathbf{\Sigma}:\|\mathbf{\Sigma}_{S(\delta_n)} - \mathbf{\Sigma}_{S(\delta_n)}^{(0)}\|_2 > M\epsilon_n/2\right\}} \mathbb{E}_{\mathbf{\Sigma}_{S(\delta_n)}}\left(1 - \phi_{S(\delta_n)}\right)
$$

$$\leq \exp\left\{C_4(\tau_n + 2s_n) - \frac{C_4 M}{8} n\epsilon_n^2\right\}.$$

The proof is thus completed. □

5.1.4 Proof of Lemma 8

Proof of Lemma 8. The proof of Lemma 8 is quite similar to that of Lemma 6, except that the following oracle test lemma for the infinity norm is applied instead of Lemma 10.

Lemma 11 *Let $\mathbf{x}_1, \ldots, \mathbf{x}_n \sim \mathrm{N}_d(\mathbf{0}_d, \boldsymbol{\Sigma})$ independently, where $\boldsymbol{\Sigma} \in \mathbb{R}^{d\times d}$. Let $\epsilon \in (0,1)$. Then there exists some absolute constant $C_6 > 0$, such that for each M satisfying $M \geq \max[4, \{(2\log 2)/C_6\}^2]$, and $M\epsilon \leq \min(1, 2\|\boldsymbol{\Sigma}_0\|_2)$, there exists a test function $\phi_n : \mathbb{R}^{n\times d} \to [0,1]$, such that*

$$\mathbb{E}_0(\phi_n) \leq 4\exp\left(4d - \frac{C_6 M^2 n\epsilon^2}{4\|\boldsymbol{\Sigma}_0\|_\infty^2}\right) + 8\exp\left(4d - \frac{C_6\sqrt{M}n}{2}\right)$$

$$\sup_{\{\|\boldsymbol{\Sigma}-\boldsymbol{\Sigma}_0\|_\infty > M\epsilon\}} \mathbb{E}_{\boldsymbol{\Sigma}}(1-\phi_n) \leq 4\exp\left\{4d - \frac{C_6 M n\epsilon^2}{4}\min\left(1, \frac{1}{4\|\boldsymbol{\Sigma}_0\|_\infty^2}\right)\right\}.$$

Let $S_0 = \mathrm{supp}(\mathbf{U}_0)$ and $S(\delta) = \mathrm{supp}_\delta(\mathbf{B})$. Then there exists some permutation matrix $\mathbf{P}$ such that

$$\mathbf{B} = \mathbf{P}\begin{bmatrix}\mathbf{B}_\delta \\ \mathbf{A}_\delta\end{bmatrix} \quad \text{and} \quad \mathbf{U}_0 = \mathbf{P}\begin{bmatrix}\mathbf{U}_{0\delta} \\ \mathbf{0}\end{bmatrix},$$

where $\mathbf{B}_\delta$ and $\mathbf{U}_{0\delta}$ are $|S(\delta)\cup S_0| \times r$ matrix. Hence for $\boldsymbol{\Sigma} \in \mathcal{G}(\delta, \tau, t)$, it holds that

$$\begin{aligned}
\|\boldsymbol{\Sigma} - \boldsymbol{\Sigma}_0\|_\infty &= \left\|\mathbf{P}\begin{bmatrix}\mathbf{B}_\delta\mathbf{B}_\delta^{\mathrm{T}} + \sigma^2\mathbf{I} - \mathbf{U}_{0\delta}\boldsymbol{\Lambda}_0\mathbf{U}_{0\delta}^{\mathrm{T}} - \sigma_0^2\mathbf{I} & \mathbf{B}_\delta\mathbf{A}_\delta^{\mathrm{T}} \\ \mathbf{A}_\delta\mathbf{B}_\delta^{\mathrm{T}} & \mathbf{A}_\delta\mathbf{A}_\delta^{\mathrm{T}} + (\sigma^2 - \sigma_0^2)\mathbf{I}_d\end{bmatrix}\mathbf{P}^{\mathrm{T}}\right\|_\infty \\
&\leq \left\|\begin{bmatrix}\mathbf{B}_\delta\mathbf{B}_\delta^{\mathrm{T}} + \sigma^2\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \sigma^2\end{bmatrix} - \begin{bmatrix}\mathbf{U}_{0\delta}\boldsymbol{\Lambda}_0\mathbf{U}_{0\delta}^{\mathrm{T}} + \sigma_0^2\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \sigma_0^2\end{bmatrix}\right\|_\infty + \left\|\begin{bmatrix}\mathbf{0} & \mathbf{B}_\delta\mathbf{A}_\delta^{\mathrm{T}} \\ \mathbf{A}_\delta\mathbf{B}_\delta^{\mathrm{T}} & \mathbf{A}_\delta\mathbf{A}_\delta^{\mathrm{T}}\end{bmatrix}\right\|_\infty \\
&\leq \left\|\boldsymbol{\Sigma}_{S(\delta)} - \boldsymbol{\Sigma}_{S(\delta)}^{(0)}\right\|_\infty + \max\left(\|\mathbf{B}_\delta\mathbf{A}_\delta^{\mathrm{T}}\|_\infty, \|\mathbf{A}_\delta\mathbf{B}_\delta^{\mathrm{T}}\|_\infty + \|\mathbf{A}_\delta\mathbf{A}_\delta^{\mathrm{T}}\|_\infty\right),
\end{aligned}$$

where

$$\boldsymbol{\Sigma}_{S(\delta)} = \begin{bmatrix}\mathbf{B}_\delta\mathbf{B}_\delta^{\mathrm{T}} + \sigma^2\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \sigma^2\end{bmatrix} \quad \text{and} \quad \boldsymbol{\Sigma}_{S(\delta)}^{(0)} = \begin{bmatrix}\mathbf{U}_{0\delta}\boldsymbol{\Lambda}_0\mathbf{U}_{0\delta}^{\mathrm{T}} + \sigma_0^2\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \sigma_0^2\end{bmatrix}.$$

Since

$$\max\left(\|\mathbf{B}_\delta\mathbf{A}_\delta^{\mathrm{T}}\|_\infty, \|\mathbf{A}_\delta\mathbf{B}_\delta^{\mathrm{T}}\|_\infty + \|\mathbf{A}_\delta\mathbf{A}_\delta^{\mathrm{T}}\|_\infty\right)$$
$$\leq \max\left(\|\mathbf{B}_\delta\|_\infty\|\mathbf{A}_\delta^{\mathrm{T}}\|_\infty, \|\mathbf{A}_\delta\|_\infty\|\mathbf{B}_\delta^{\mathrm{T}}\|_\infty + \|\mathbf{A}_\delta\|_\infty\|\mathbf{A}_\delta^{\mathrm{T}}\|_\infty\right),$$

and

$$\begin{aligned}
\|\mathbf{B}_\delta\|_\infty &= \max_{j\in S_0\cup S(\delta)}\|\mathbf{B}_{j*}\|_1 \leq \sum_{j=1}^p \|\mathbf{B}_{j*}\|_1\mathbb{1}\{j\in S(\delta)\cup S_0\} \leq t,\\
\|\mathbf{B}_\delta^{\mathrm{T}}\|_\infty &\leq \sum_{j=1}^p \|\mathbf{B}_{j*}\|_1\mathbb{1}\{j\in S(\delta)\cup S_0\} \leq t,\\
\|\mathbf{A}_\delta\|_\infty &= \max_{j\in S_0^c\cap S(\delta)^c}\|\mathbf{B}_{j*}\|_1 \leq \max_{j\in S(\delta)^c}\|\mathbf{B}_{j*}\|_1 \leq \delta,\\
\|\mathbf{A}_\delta^{\mathrm{T}}\|_\infty &\leq \sum_{j=1}^p \|\mathbf{B}_{j*}\|_1\mathbb{1}\left\{j\in S_0^c\cap S(\delta)^c\right\} \leq \sum_{j=1}^p \|\mathbf{B}_{j*}\|_1\mathbb{1}\left\{j\in S(\delta)^c\right\} \leq p\delta,
\end{aligned}$$

it follows that

$$\|\mathbf{\Sigma}-\mathbf{\Sigma}_0\|_\infty \leq \left\|\mathbf{\Sigma}_{S(\delta)} - \mathbf{\Sigma}_{S(\delta)}^{(0)}\right\|_\infty + \max(p\delta_n t_n, \delta_n t_n + p\delta_n^2) \leq \left\|\mathbf{\Sigma}_{S(\delta)} - \mathbf{\Sigma}_{S(\delta)}^{(0)}\right\|_\infty + M_1\epsilon_n.$$

By taking $M \geq 2M_1$, we obtain

$$\begin{aligned}
&\left\{\mathbf{\Sigma} = \mathbf{B}\mathbf{B}^{\mathrm{T}} + \sigma^2\mathbf{I} : \|\mathbf{\Sigma}-\mathbf{\Sigma}_0\|_\infty > M\epsilon_n, \mathbf{B}\in\mathcal{G}(\delta_n,\tau_n,t_n)\right\}\\
&\quad\subset \left\{\mathbf{\Sigma} : \left\|\mathbf{\Sigma}_{S(\delta)} - \mathbf{\Sigma}_{S(\delta)}^{(0)}\right\|_\infty > \frac{M}{2}\epsilon_n : \mathbf{B}\in\mathcal{G}(\delta_n,\tau_n,t_n)\right\}\\
&\quad\subset \bigcup_{S(\delta_n)\subset[p]:|S(\delta)|\leq\tau_n}\left\{\mathbf{\Sigma} : \left\|\mathbf{\Sigma}_{S(\delta)} - \mathbf{\Sigma}_{S(\delta)}^{(0)}\right\|_\infty > \frac{M}{2}\epsilon_n\right\}.
\end{aligned}$$

Since both $\mathbf{\Sigma}_{S(\delta_n)}$ and $\mathbf{\Sigma}_{S(\delta_n)}^{(0)}$ are $(|S(\delta_n)\cup S_0|+1)\times(|S(\delta_n)\cup S_0|+1)$ square matrices, and

$$|S(\delta_n)\cup S_0| + 1 \leq |S(\delta_n)| + S_0 + 1 \leq \tau_n + 2s_n,$$

then for each $S(\delta_n)\subset[p]$ with $|S(\delta_n)|\leq\tau_n$, and for each

$$M \in \left[\max\left\{\frac{M_1}{2}, 8, \frac{8(\log 2)^2}{C_6^2}\right\}, \frac{2\min(1, 2\|\mathbf{\Sigma}_0\|_2)}{\epsilon_n}\right],$$

we invoke Lemma 11 to construct a test $\phi_{S(\delta_n)}$ depending on the index set $S(\delta_n)$, such that the type I error probability satisfies

$$\mathbb{E}_{\mathbf{\Sigma}_{S(\delta_n)}^{(0)}}\left(\phi_{S(\delta_n)}\right) \leq 4\exp\left\{4(\tau_n+2s_n) - \frac{C_6M^2n\epsilon_n^2}{16\|\mathbf{\Sigma}_{S(\delta_n)}^{(0)}\|_\infty^2}\right\} \tag{5.12}$$

$$+ 8\exp\left\{4(\tau_n + 2s_n) - C_6\sqrt{\frac{M}{2}n}\right\}$$
$$\leq 12\exp\left\{4(\tau_n + 2s_n) - C_6\min\left(\frac{M^2}{16\|\mathbf{\Sigma}_0\|_\infty^2}, \sqrt{\frac{M}{2}}\right)n\epsilon_n^2\right\}$$
$$\leq 12\exp\left\{4(\tau_n + 2s_n) - C_6\min\left(\frac{1}{2}, \frac{\|\mathbf{\Sigma}_0\|_\infty^2}{\sqrt{2}}\right)\frac{\sqrt{M}n\epsilon_n^2}{\|\mathbf{\Sigma}_0\|_\infty^2}\right\}.$$

In addition, for all $\mathbf{\Sigma}_{S(\delta_n)} \in \{\|\mathbf{\Sigma}_{S(\delta_n)} - \mathbf{\Sigma}^{(0)}_{S(\delta_n)}\|_2 > M\epsilon_n/2\}$, the type II error probability satisfies

$$\mathbb{E}_{\mathbf{\Sigma}^{(1)}_{S(\delta_n)}}\left(1 - \phi_{S(\delta_n)}\right) \leq 4\exp\left\{4(\tau_n + 2s_n) - \frac{C_6 M n\epsilon_n^2}{8}\min\left(1, \frac{1}{4\|\mathbf{\Sigma}_0\|_\infty^2}\right)\right\}$$
$$\leq 4\exp\left\{4(\tau_n + 2s_n) - C_6\min\left(\frac{\|\mathbf{\Sigma}_0\|_\infty^2}{8}, \frac{1}{32}\right)\frac{Mn\epsilon_n^2}{\|\mathbf{\Sigma}_0\|_\infty^2}\right\}.$$

Notice that for each index set $S(\delta_n)$, the test function $\phi_{S(\delta_n)}$ is only a function of $\mathbf{Y}_n$ through the coordinates $[A_{ij} : i \in [n], j \in S(\delta_n) \cup S_0]$. Hence, $\mathbb{E}_{\mathbf{\Sigma}^{(0)}_{S(\delta_n)}}(\phi_{S(\delta_n)}) = \mathbb{E}_0(\phi_{S(\delta_n)})$, and for any $p \times p$ covariance matrix $\mathbf{\Sigma}$ with $\|\mathbf{\Sigma}_{S(\delta_n)} - \mathbf{\Sigma}^{(0)}_{S(\delta_n)}\|_\infty > M\epsilon_n/2$, it holds that $\mathbb{E}_{\mathbf{\Sigma}_{S(\delta_n)}}(1 - \phi_{S(\delta_n)}) = \mathbb{E}_{\mathbf{\Sigma}}(1 - \phi_{S(\delta_n)})$. Therefore, by aggregating the test functions

$$\phi_n = \max_{S(\delta_n)\subset[p]:|S(\delta_n)|\leq\tau_n} \phi_{S(\delta_n)},$$

we obtain

$$\mathbb{E}_0(\phi_n) \leq \sum_{S(\delta_n)\subset[p]:|S(\delta_n)|\leq\tau_n} \mathbb{E}_{\mathbf{\Sigma}^{(0)}_{S(\delta_n)}}(\phi_{S(\delta_n)})$$
$$\leq 12\sum_{q=0}^{\lfloor\tau_n\rfloor}\frac{p!}{q!(p-q)!}\exp\left\{4(\tau_n + 2s_n) - C_6\min\left(\frac{1}{2}, \frac{\|\mathbf{\Sigma}_0\|_\infty^2}{\sqrt{2}}\right)\frac{\sqrt{M}n\epsilon_n^2}{\|\mathbf{\Sigma}_0\|_\infty^2}\right\}$$
$$\leq 12(\tau_n + 1)\exp(\tau_n\log p)\exp\left\{4(\tau_n + 2s_n) - C_6\min\left(\frac{1}{2}, \frac{\|\mathbf{\Sigma}_0\|_\infty^2}{\sqrt{2}}\right)\frac{\sqrt{M}n\epsilon_n^2}{\|\mathbf{\Sigma}_0\|_\infty^2}\right\}$$
$$\leq 12\exp\left\{\tau_n + \tau_n\log p + 4(\tau_n + 2s_n) - C_6\min\left(\frac{1}{2}, \frac{\|\mathbf{\Sigma}_0\|_\infty^2}{\sqrt{2}}\right)\frac{\sqrt{M}n\epsilon_n^2}{\|\mathbf{\Sigma}_0\|_\infty^2}\right\}$$
$$\leq 12\exp\left\{6(\tau_n\log p + 2s_n) - C_6\min\left(\frac{1}{2}, \frac{\|\mathbf{\Sigma}_0\|_\infty^2}{\sqrt{2}}\right)\frac{\sqrt{M}n\epsilon_n^2}{\|\mathbf{\Sigma}_0\|_\infty^2}\right\},$$

and

$$\sup_{\mathbf{\Sigma}\in H_1}\mathbb{E}_{\mathbf{\Sigma}}(1 - \phi_n) \leq \sup_{S(\delta_n)\subset[p]:|S(\delta_n)|\leq\tau_n}\ \sup_{\left\{\mathbf{\Sigma}:\|\mathbf{\Sigma}_{S(\delta_n)} - \mathbf{\Sigma}^{(0)}_{S(\delta_n)}\|_2 > M\epsilon_n/2\right\}}\mathbb{E}_{\mathbf{\Sigma}_{S(\delta_n)}}\left(1 - \phi_{S(\delta_n)}\right)$$

$$\leq 4\exp\left\{4(\tau_n + 2s_n) - C_6 \min\left(\frac{\|\boldsymbol{\Sigma}_0\|_\infty^2}{8}, \frac{1}{32}\right)\frac{Mn\epsilon_n^2}{\|\boldsymbol{\Sigma}_0\|_\infty^2}\right\}.$$

The proof is thus completed. □

5.1.5 Additional technical results and proofs

Proof of Lemma 7. Since $p(w) = (1-\theta)\lambda_0 e^{-\lambda_0 w} + \theta\{\lambda_0^r/\Gamma(r)\}w^{r-1}e^{-\lambda w}$, then

$$\mathbb{P}(\xi = 1) = (1-\theta)\int_\delta^\infty \lambda_0 e^{-\lambda_0 w}\mathrm{d}w + \theta\int_\delta^\infty \frac{\lambda^r}{\Gamma(r)}w^{r-1}e^{-\lambda w}\mathrm{d}w = (1-\theta)e^{-\lambda_0\delta} + \theta\frac{\Gamma(r,\lambda\delta)}{\Gamma(r)}.$$

Then for any measurable $A \subset \mathbb{R}$, we have

$$\begin{aligned}
&\mathbb{P}(w \in A \mid \xi = 1)\\
&\quad= \frac{1}{\mathbb{P}(\xi = 1)}\left\{(1-\theta)\int_A \mathbb{1}(w > \delta)\lambda_0 e^{-\lambda_0 w}\mathrm{d}w + \theta\int_A \mathbb{1}(w > \delta)\frac{\lambda^r}{\Gamma(r)}w^{r-1}e^{-\lambda w}\mathrm{d}w\right\}\\
&\quad= \int_A \mathbb{1}(w > \delta)\left\{(1-\theta')\lambda_0 e^{-\lambda_0(w-\delta)}\mathrm{d}w + \theta'\frac{\lambda^r}{\Gamma(r,\lambda\delta)}w^{r-1}e^{-\lambda w}\right\}\mathrm{d}w,
\end{aligned}$$

where

$$\theta' = \frac{\theta\Gamma(r,\lambda\delta)/\Gamma(r)}{(1-\theta)e^{-\lambda_0\delta} + \theta\Gamma(r,\lambda\delta)/\Gamma(r)} \in (0,1).$$

Therefore,

$$p(w \mid \xi = 1) = \left\{(1-\theta')\lambda_0 e^{-\lambda_0(w-\delta)}\mathrm{d}w + \theta'\frac{\lambda^r}{\Gamma(r,\lambda\delta)}w^{r-1}e^{-\lambda w}\right\}\mathbb{1}(w > \delta).$$

Hence we proceed and compute

$$\begin{aligned}
&\{E(w^m \mid \xi = 1)\}^{1/m}\\
&\quad= \left\{(1-\theta')\int_\delta^\infty w^m\lambda_0 e^{-\lambda_0(w-\delta)}\mathrm{d}w + \theta'\frac{\Gamma(r)}{\Gamma(r,\lambda\delta)}\int_\delta^\infty w^m\frac{\lambda^r}{\Gamma(r)}w^{r-1}e^{-\lambda w}\mathrm{d}w\right\}^{1/m}\\
&\quad\leq \left\{\int_0^\infty (w+\delta)^m\lambda_0 e^{-\lambda_0 w}\mathrm{d}w + \frac{\Gamma(r)}{\Gamma(r,\lambda\delta)}\int_0^\infty w^m\frac{\lambda^r}{\Gamma(r)}w^{r-1}e^{-\lambda w}\mathrm{d}w\right\}^{1/m}\\
&\quad= \left\{\int_0^\delta (w+\delta)^m\lambda_0 e^{-\lambda_0 w}\mathrm{d}w + \int_\delta^\infty (w+\delta)^m\lambda_0 e^{-\lambda_0 w}\mathrm{d}w + \frac{\Gamma(r)}{\Gamma(r,\lambda\delta)}\frac{(r+m-1)!}{(r-1)!\lambda^m}\right\}^{1/m}\\
&\quad\leq \left\{\int_0^\infty (2\delta)^m\lambda_0 e^{-\lambda_0 w}\mathrm{d}w + \int_0^\infty (2w)^m\lambda_0 e^{-\lambda_0 w}\mathrm{d}w + \frac{\Gamma(r)}{\Gamma(r,\lambda\delta)}\frac{(r+m-1)!}{(r-1)!\lambda^m}\right\}^{1/m}
\end{aligned}$$

$$= \left\{ (2\delta)^m + 2^m \frac{m!}{\lambda_0^m} + \frac{\Gamma(r)}{\Gamma(r, \lambda\delta)} \frac{(r+m-1)!}{(r-1)!\lambda^m} \right\}^{1/m} \leq 2\delta + \frac{2m}{\lambda_0} + \frac{2(r+m)}{\lambda}.$$

Hence

$$\sup_{m\geq 1} \{E(w^m)\}^{1/m} \leq \sup_{m\geq 1} \frac{1}{m} \left\{ 2\delta + \frac{2m}{\lambda_0} + \frac{2(r+m)}{\lambda} \right\} = 2\delta + \frac{2}{\lambda_0} + \frac{2(r+1)}{\lambda}.$$

Now we compute the sub-exponential norm. Write

$$\begin{aligned}
\sup_{m\geq 1} \frac{1}{m} \{E(w^m)\}^{1/m} &= \sup_{m\geq 1} \frac{1}{m} \left\{ (1-\theta) \int_0^\infty w^m \lambda_0 \mathrm{e}^{-\lambda_0 w} \mathrm{d}w + \theta \int_0^\infty w^m \frac{\lambda^r}{\Gamma(r)} w^{r-1} \mathrm{e}^{-\lambda w} \mathrm{d}w \right\}^{1/m} \\
&= \sup_{m\geq 1} \frac{1}{m} \left\{ (1-\theta) \frac{m!}{\lambda_0^m} + \theta \frac{(m+r-1)!}{\lambda^m (r-1)!} \right\}^{1/m} \\
&\leq \frac{1}{\lambda_0} + \frac{1}{\lambda} \sup_{m\geq 1} \theta^{1/m} \left(1 + \frac{r}{m}\right) \leq \frac{1}{\lambda_0} + \frac{r+1}{\lambda}.
\end{aligned}$$

If $\theta \leq \mathrm{e}^{-r}$, we can further derive the following result using the fact that $\log(1+ru) \leq ru$ for $u \in (0, 1]$:

$$\begin{aligned}
\sup_{m\geq 1} \frac{1}{m} \{E(w^m)\}^{1/m} &\leq \frac{1}{\lambda_0} + \frac{1}{\lambda} \sup_{m\geq 1} \theta^{1/m} \left(1 + \frac{r}{m}\right) \\
&\leq \frac{1}{\lambda_0} + \frac{1}{\lambda} \exp\left[\sup_{u\in(0,1]} \{-ru + \log(1+ru)\} \right] \\
&\leq \frac{1}{\lambda_0} + \frac{1}{\lambda}.
\end{aligned}$$

□

Proof of Lemma 11. Denote the alternative set by $\mathcal{H}_1 = \{\boldsymbol{\Sigma} : \|\boldsymbol{\Sigma} - \boldsymbol{\Sigma}_0\|_\infty > M\epsilon\}$ and decompose it as follows: $\mathcal{H}_1 = \bigcup_{j=0}^\infty \mathcal{H}_{1j}$, where

$$\begin{aligned}
\mathcal{H}_{10} &= \left\{ \|\boldsymbol{\Sigma} - \boldsymbol{\Sigma}_0\|_\infty > M\epsilon, \|\boldsymbol{\Sigma}\|_\infty \leq (\sqrt{M}+2)\|\boldsymbol{\Sigma}_0\|_\infty \right\} \\
\mathcal{H}_{1j} &= \left\{ (\sqrt{M}+2)(M\epsilon^2)^{-(j-1)/2}\|\boldsymbol{\Sigma}_0\|_\infty < \|\boldsymbol{\Sigma}\|_\infty \leq (\sqrt{M}+2)(M\epsilon^2)^{-j/2}\|\boldsymbol{\Sigma}_0\|_\infty \right\}.
\end{aligned}$$

For each $\mathcal{H}_{1j}$, we construct test functions ϕ_{nj} as follows:

$$\begin{aligned}
\phi_{n0} &= \mathbb{1}\left\{ \left\| \frac{1}{n} \sum_{i=1}^n \mathbf{x}_i \mathbf{x}_i^{\mathrm{T}} - \boldsymbol{\Sigma}_0 \right\|_\infty > M\epsilon/2 \right\}, \\
\phi_{nj} &= \mathbb{1}\left\{ \left\| \frac{1}{n} \sum_{i=1}^n \mathbf{x}_i \mathbf{x}_i^{\mathrm{T}} \right\|_\infty > \frac{\sqrt{M}+2}{2} \|\boldsymbol{\Sigma}_0\|_\infty (M\epsilon^2)^{-(j-1)/2} \right\}.
\end{aligned}$$

We first control the type I error. By Lemma 12 presented later,

$$\mathbb{E}_0(\phi_{n0}) \le 4\exp\left\{4d - C_6 n \min\left(\frac{M\epsilon}{2\|\boldsymbol{\Sigma}_0\|_\infty}, \frac{M^2\epsilon^2}{4\|\boldsymbol{\Sigma}_0\|_\infty^2}\right)\right\} \le 4\exp\left(4d - \frac{C_6 M^2 n\epsilon^2}{4\|\boldsymbol{\Sigma}_0\|_\infty^2}\right)$$

since $M\epsilon < 2\|\boldsymbol{\Sigma}_0\|_\infty$ by assumption. In addition, $M\epsilon^2 \le \sqrt{M}M\epsilon^2 \le (M\epsilon)^2 \le 1$, and hence, for any $j \ge 1$,

$$\begin{aligned}
\mathbb{E}_0(\phi_{nj}) &\le \mathbb{P}_0\left\{\left\|\frac{1}{n}\sum_{i=1}^n \mathbf{x}_i\mathbf{x}_i^{\mathrm{T}} - \boldsymbol{\Sigma}_0\right\|_\infty + \|\boldsymbol{\Sigma}_0\|_\infty > \frac{\sqrt{M}+2}{2}\|\boldsymbol{\Sigma}_0\|_\infty (M\epsilon^2)^{-(j-1)/2}\right\} \\
&\le \mathbb{P}_0\left\{\left\|\frac{1}{n}\sum_{i=1}^n \mathbf{x}_i\mathbf{x}_i^{\mathrm{T}} - \boldsymbol{\Sigma}_0\right\|_\infty > \frac{\sqrt{M}}{2}\|\boldsymbol{\Sigma}_0\|_\infty (M\epsilon^2)^{-(j-1)/2}\right\} \\
&\le 4\exp\left[4d - C_6 n \min\left\{\frac{M(M\epsilon^2)^{2-j}}{4}, \frac{\sqrt{M}(M\epsilon^2)^{1/2-j/2}}{2}\right\}\right] \\
&\le 4\exp\left(4d - C_6\frac{M^{1-j/2}n\epsilon^{-(j-1)}}{2}\right).
\end{aligned}$$

Next we consider the type II error. For any $\boldsymbol{\Sigma} \in \mathcal{H}_{10}$, the type II error probability can be upper bounded by

$$\begin{aligned}
\mathbb{E}_{\boldsymbol{\Sigma}}(1-\phi_{n0}) &\le \mathbb{P}_{\boldsymbol{\Sigma}}\left\{\|\boldsymbol{\Sigma} - \boldsymbol{\Sigma}_0\|_\infty - \left\|\frac{1}{n}\sum_{i=1}^n \mathbf{x}_i\mathbf{x}_i^{\mathrm{T}} - \boldsymbol{\Sigma}\right\|_\infty \le M\epsilon/2\right\} \\
&\le \mathbb{P}_{\boldsymbol{\Sigma}}\left\{\left\|\frac{1}{n}\sum_{i=1}^n \mathbf{x}_i\mathbf{x}_i^{\mathrm{T}} - \boldsymbol{\Sigma}\right\|_\infty > M\epsilon/2\right\} \\
&\le \mathbb{P}_{\boldsymbol{\Sigma}}\left\{\left\|\frac{1}{n}\sum_{i=1}^n \mathbf{x}_i\mathbf{x}_i^{\mathrm{T}} - \boldsymbol{\Sigma}\right\|_\infty > \|\boldsymbol{\Sigma}\|_\infty \frac{M\epsilon}{2(\sqrt{M}+2)\|\boldsymbol{\Sigma}_0\|_\infty}\right\} \\
&\le 4\exp\left\{4d - \frac{C_6 M^2 n\epsilon^2}{4(\sqrt{M}+2)^2\|\boldsymbol{\Sigma}_0\|_\infty^2}\right\},
\end{aligned}$$

where the last inequality is due to Lemma 12 and the assumption $M\epsilon < 2\|\boldsymbol{\Sigma}_0\|_\infty$. For any $\boldsymbol{\Sigma} \in \mathcal{H}_{1j}$ with $j \ge 1$, we estimate the type II error as follows:

$$\begin{aligned}
\mathbb{E}_{\boldsymbol{\Sigma}}(1-\phi_{nj}) &\le \mathbb{P}_{\boldsymbol{\Sigma}}\left\{\|\boldsymbol{\Sigma}\|_\infty - \left\|\frac{1}{n}\sum_{i=1}^n \mathbf{x}_i\mathbf{x}_i^{\mathrm{T}} - \boldsymbol{\Sigma}\right\|_\infty \le \frac{\sqrt{M}+2}{2}\|\boldsymbol{\Sigma}_0\|_\infty (M\epsilon^2)^{-(j-1)/2}\right\} \\
&\le \mathbb{P}_{\boldsymbol{\Sigma}}\left\{\left\|\frac{1}{n}\sum_{i=1}^n \mathbf{x}_i\mathbf{x}_i^{\mathrm{T}} - \boldsymbol{\Sigma}\right\|_\infty > \frac{\sqrt{M}+2}{2}\|\boldsymbol{\Sigma}_0\|_\infty (M\epsilon^2)^{-(j-1)/2}\right\} \\
&= \mathbb{P}_{\boldsymbol{\Sigma}}\left\{\left\|\frac{1}{n}\sum_{i=1}^n \mathbf{x}_i\mathbf{x}_i^{\mathrm{T}} - \boldsymbol{\Sigma}\right\|_\infty > \frac{(M\epsilon^2)^{1/2}}{2}(\sqrt{M}+2)(M\epsilon^2)^{-j/2}\|\boldsymbol{\Sigma}_0\|_\infty\right\} \\
&\le \mathbb{P}_{\boldsymbol{\Sigma}}\left\{\left\|\frac{1}{n}\sum_{i=1}^n \mathbf{x}_i\mathbf{x}_i^{\mathrm{T}} - \boldsymbol{\Sigma}\right\|_\infty > \frac{(M\epsilon^2)^{1/2}}{2}\|\boldsymbol{\Sigma}\|_\infty\right\}
\end{aligned}$$

$$\leq 4\exp\left(4d - \frac{C_6 M n\epsilon^2}{4}\right)$$

since $M\epsilon^2 \leq M\epsilon \leq 1$. Now we aggregate the individual tests by taking $\phi_n = \sup_{j\geq 0}\phi_{nj}$. Then the overall type I error probability can be bounded by

$$\begin{aligned}
\mathbb{E}_0(\phi_n) &\leq \sum_{j=0}^{\infty}\mathbb{E}_0(\phi_{nj}) \\
&\leq 4\exp\left(4d - \frac{C_6 M^2 n\epsilon^2}{4\|\boldsymbol{\Sigma}_0\|_\infty^2}\right) + \sum_{j=1}^{\infty} 4\exp\left(4d - C_6\frac{M^{1-j/2} n\epsilon^{-(j-1)}}{2}\right) \\
&= 4\exp\left(4d - \frac{C_6 M^2 n\epsilon^2}{4\|\boldsymbol{\Sigma}_0\|_\infty^2}\right) + 4\exp(4d)\sum_{j=1}^{\infty}\exp\left\{-\frac{C_6 M n\epsilon}{2}\left(\frac{1}{\sqrt{M}\epsilon}\right)^j\right\} \\
&\leq 4\exp\left(4d - \frac{C_6 M^2 n\epsilon^2}{4\|\boldsymbol{\Sigma}_0\|_\infty^2}\right) + 4\exp(4d)\sum_{j=1}^{\infty}\exp\left\{-j\frac{C_6 M n\epsilon}{2}\left(\frac{1}{\sqrt{M}\epsilon}\right)\right\} \\
&= 4\exp\left(4d - \frac{C_6 M^2 n\epsilon^2}{4\|\boldsymbol{\Sigma}_0\|_\infty^2}\right) + 8\exp\left(4d - \frac{C_6\sqrt{M}n}{2}\right),
\end{aligned}$$

since $M \geq \{(2\log 2)/C_6\}^2$, where the simple inequality $x^j \geq jx$ for all $x \geq 1$ is applied. Furthermore, the overall type II error probability can also be bounded:

$$\begin{aligned}
\sup_{\boldsymbol{\Sigma}\in\mathcal{H}_1}\mathbb{E}_{\boldsymbol{\Sigma}}(1-\phi_n) &= \sup_{j\geq 0}\sup_{\boldsymbol{\Sigma}\in\mathcal{H}_{1j}}\mathbb{E}_{\boldsymbol{\Sigma}}(1-\phi_n) = \sup_{j\geq 0}\sup_{\boldsymbol{\Sigma}\in\mathcal{H}_{1j}}\mathbb{E}_{\boldsymbol{\Sigma}}\inf_{j\geq 0}(1-\phi_{jn}) \\
&\leq \sup_{j\geq 0}\sup_{\boldsymbol{\Sigma}\in\mathcal{H}_{1j}}\mathbb{E}_{\boldsymbol{\Sigma}}(1-\phi_{jn}) \\
&\leq \sup_{j\geq 0}\sup_{\boldsymbol{\Sigma}\in\mathcal{H}_{1j}} 4\exp\left[4d - \frac{C_6 M n\epsilon^2}{4}\min\left\{1, \frac{M}{(\sqrt{M}+2)^2\|\boldsymbol{\Sigma}_0\|_\infty^2}\right\}\right] \\
&\leq 4\exp\left\{4d - \frac{C_6 M n\epsilon^2}{4}\min\left(1, \frac{1}{4\|\boldsymbol{\Sigma}_0\|_\infty^2}\right)\right\}
\end{aligned}$$

since $M \geq 4$. The proof is thus completed. $\square$

Lemma 12 *Let $\mathbf{x}_1,\dots,\mathbf{x}_n \sim \mathrm{N}_d(\mathbf{0}_d,\boldsymbol{\Sigma})$ independently, where $\boldsymbol{\Sigma}\in\mathbb{R}^{d\times d}$. Then there exists an absolute constant $C_6 > 0$, such that for any $t > 0$,*

$$\mathbb{P}\left(\left\|\frac{1}{n}\sum_{i=1}^n \mathbf{x}_i\mathbf{x}_i^{\mathrm{T}} - \boldsymbol{\Sigma}\right\|_\infty > t\|\boldsymbol{\Sigma}\|_\infty\right) \leq 4\exp\{4d - C_6 n\min(t, t^2)\}$$

Proof of Lemma 12. By definition,

$$\|\mathbf{A}\|_\infty = \sup_{\|\mathbf{v}\|_\infty=1}\|\mathbf{A}\mathbf{v}\|_\infty = \max_{j\in[p]}\sup_{\|\mathbf{v}\|_\infty=1}\mathbf{e}_j^{\mathrm{T}}\mathbf{A}\mathbf{v},$$

where $\mathbf{e}_j$ is the unit vector along the jth coordinate direction. Now let $S_\infty^{d-1}(1/2)$ be an 1/2-net of the ℓ_∞-sphere in $\mathbb{R}^d$ ($\{\mathbf{v} \in \mathbb{R}^d : \|\mathbf{v}\|_\infty = 1\}$) with minimum cardinality. Then for each $\mathbf{v}$ with $\|\mathbf{v}\|_\infty = 1$, there exists some $\mathbf{v}' \in S_\infty^{d-1}(1/2)$ such that $\|\mathbf{v} - \mathbf{v}'\|_\infty < 1/2$. Therefore,

$$\begin{aligned}
\|\mathbf{A}\|_\infty &= \max_{j\in[d]} \sup_{\|\mathbf{v}\|_\infty=1} \mathbf{e}_j^{\mathrm{T}}\mathbf{A}\mathbf{v} \le \max_{j\in[d]} \sup_{\|\mathbf{v}\|_\infty=1} \left\{\mathbf{e}_j^{\mathrm{T}}\mathbf{A}(\mathbf{v}-\mathbf{v}') + \mathbf{e}_j^{\mathrm{T}}\mathbf{A}\mathbf{v}'\right\} \\
&\le \max_{j\in[d]} \sup_{\|\mathbf{v}\|_\infty=1} \mathbf{e}_j^{\mathrm{T}}\mathbf{A}(\mathbf{v}-\mathbf{v}') + \max_{j\in[d]} \sup_{\mathbf{v}\in S_\infty^{d-1}(1/2)} \mathbf{e}_j^{\mathrm{T}}\mathbf{A}\mathbf{v} \\
&\le \frac{1}{2}\|\mathbf{A}\|_\infty + \max_{j\in[d]} \sup_{\mathbf{v}\in S_\infty^{d-1}(1/2)} \mathbf{e}_j^{\mathrm{T}}\mathbf{A}\mathbf{v},
\end{aligned}$$

and hence,

$$\|\mathbf{A}\|_\infty \le 2\max_{j\in[d]} \sup_{\mathbf{v}\in S_\infty^{d-1}(1/2)} \mathbf{e}_j^{\mathrm{T}}\mathbf{A}\mathbf{v}.$$

Denote

$$\mathbf{E} = \frac{1}{n}\sum_{i=1}^n \mathbf{x}_i\mathbf{x}_i^{\mathrm{T}} - \mathbf{\Sigma}.$$

Now we apply the union bound to derive

$$\begin{aligned}
&\mathbb{P}\left(\left\|\frac{1}{n}\sum_{i=1}^n \mathbf{x}_i\mathbf{x}_i^{\mathrm{T}} - \mathbf{\Sigma}\right\|_\infty > t\|\mathbf{\Sigma}\|_\infty\right) \\
&= \mathbb{P}\left[\bigcup_{j\in[d]} \bigcup_{\mathbf{v}\in S_\infty^{d-1}(1/2)} \left\{\mathbf{e}_j^{\mathrm{T}}\mathbf{E}\mathbf{v} > \frac{t}{2}\|\mathbf{\Sigma}\|_\infty\right\}\right] \\
&\le \sum_{j=1}^d \sum_{\mathbf{v}\in S_\infty^{d-1}(1/2)} \mathbb{P}\left\{\mathbf{e}_j^{\mathrm{T}}\left(\frac{1}{n}\sum_{i=1}^n \mathbf{x}_i\mathbf{x}_i^{\mathrm{T}} - \mathbf{\Sigma}\right)\mathbf{v} > \frac{t}{2}\|\mathbf{\Sigma}\|_\infty\right\} \\
&= \sum_{j=1}^d \sum_{\mathbf{v}\in S_\infty^{d-1}(1/2)} \mathbb{P}\left\{\frac{1}{n}\sum_{i=1}^n (\mathbf{e}_j^{\mathrm{T}}\mathbf{x}_i)(\mathbf{v}^{\mathrm{T}}\mathbf{x}_i) - \mathbf{e}_j^{\mathrm{T}}\mathbf{\Sigma}\mathbf{v} > \frac{t}{2}\|\mathbf{\Sigma}\|_\infty\right\}.
\end{aligned}$$

Observe that

$$\begin{bmatrix}\mathbf{v}^{\mathrm{T}}\mathbf{x}_i \\ \mathbf{e}_j^{\mathrm{T}}\mathbf{x}_i\end{bmatrix} \sim \mathrm{N}_2\left(\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}\mathbf{v}^{\mathrm{T}}\mathbf{\Sigma}\mathbf{v} & \mathbf{v}^{\mathrm{T}}\mathbf{\Sigma}\mathbf{e}_j \\ \mathbf{e}_j^{\mathrm{T}}\mathbf{\Sigma}\mathbf{v} & \mathbf{e}_j^{\mathrm{T}}\mathbf{\Sigma}\mathbf{e}_j\end{bmatrix}\right),$$

then we can decompose $(\mathbf{e}_j^{\mathrm{T}}\mathbf{x}_i)(\mathbf{v}^{\mathrm{T}}\mathbf{x}_i)$ by projecting $\mathbf{v}^{\mathrm{T}}\mathbf{x}_i$ onto the space spanned by $\mathbf{e}_j^{\mathrm{T}}\mathbf{x}_i$ as follows:

$$(\mathbf{e}_j^{\mathrm{T}}\mathbf{x}_i)(\mathbf{v}^{\mathrm{T}}\mathbf{x}_i) = \left(\mathbf{v}^{\mathrm{T}}\mathbf{x}_i - \frac{\mathbf{e}_j^{\mathrm{T}}\mathbf{\Sigma}\mathbf{v}}{\mathbf{e}_j^{\mathrm{T}}\mathbf{\Sigma}\mathbf{e}_j}\mathbf{e}_j^{\mathrm{T}}\mathbf{x}_i\right)(\mathbf{e}_j^{\mathrm{T}}\mathbf{x}_i) + \frac{\mathbf{e}_j^{\mathrm{T}}\mathbf{\Sigma}\mathbf{v}}{\mathbf{e}_j^{\mathrm{T}}\mathbf{\Sigma}\mathbf{e}_j}(\mathbf{e}_j^{\mathrm{T}}\mathbf{x}_i)^2$$

$$\overset{d}{=} \sqrt{\mathbf{e}_j^{\mathrm{T}}\boldsymbol{\Sigma}\mathbf{e}_j\mathbf{v}^{\mathrm{T}}\boldsymbol{\Sigma}\mathbf{v} - (\mathbf{e}_j^{\mathrm{T}}\boldsymbol{\Sigma}\mathbf{v})^2}\zeta_{i1}\zeta_{i2} + \mathbf{e}_j^{\mathrm{T}}\boldsymbol{\Sigma}\mathbf{v}\zeta_{i2}^2,$$

where ζ_{i1} and ζ_{i2} are independent $\mathrm{N}(0,1)$ random variables, $i = 1,\dots,n$, and $\overset{d}{=}$ indicates the equality in distribution. Hence,

$$\begin{aligned}
&\mathbb{P}\left\{\frac{1}{n}\sum_{i=1}^n(\mathbf{e}_j^{\mathrm{T}}\mathbf{x}_i)(\mathbf{v}^{\mathrm{T}}\mathbf{x}_i) - \mathbf{e}_j^{\mathrm{T}}\boldsymbol{\Sigma}\mathbf{v} > \frac{t}{2}\|\boldsymbol{\Sigma}\|_\infty\right\}\\
&\quad\le \mathbb{P}\left\{\sqrt{\mathbf{e}_j^{\mathrm{T}}\boldsymbol{\Sigma}\mathbf{e}_j\mathbf{v}^{\mathrm{T}}\boldsymbol{\Sigma}\mathbf{v} - (\mathbf{e}_j^{\mathrm{T}}\boldsymbol{\Sigma}\mathbf{v})^2}\left|\frac{1}{n}\sum_{i=1}^n\zeta_{i1}\zeta_{i2}\right| + |\mathbf{e}_j^{\mathrm{T}}\boldsymbol{\Sigma}\mathbf{v}|\left|\frac{1}{n}\sum_{i=1}^n\left(\zeta_{i2}^2-1\right)\right| > \frac{t}{2}\|\boldsymbol{\Sigma}\|_\infty\right\}\\
&\quad\le \mathbb{P}\left\{\|\boldsymbol{\Sigma}\|_\infty\left|\frac{1}{n}\sum_{i=1}^n\zeta_{i1}\zeta_{i2}\right| + \|\boldsymbol{\Sigma}\|_\infty\left|\frac{1}{n}\sum_{i=1}^n\left(\zeta_{i2}^2-1\right)\right| > \frac{t}{2}\|\boldsymbol{\Sigma}\|_\infty\right\}\\
&\quad\le \mathbb{P}\left\{\left|\frac{1}{n}\sum_{i=1}^n\zeta_{i1}\zeta_{i2}\right| > \frac{t}{4}\right\} + \mathbb{P}\left\{\left|\frac{1}{n}\sum_{i=1}^n\left(\zeta_{i2}^2-1\right)\right| > \frac{t}{4}\right\}.
\end{aligned}$$

Since $\zeta_{i1}\zeta_{i2}$ and $\zeta_{i2}^2 - 1$ are mean-zero sub-exponential random variables, it follows from the large-deviation inequality for sub-exponential random variables that

$$\mathbb{P}\left\{\left|\frac{1}{n}\sum_{i=1}^n\zeta_{i1}\zeta_{i2}\right| > \frac{t}{4}\right\} + \mathbb{P}\left\{\left|\frac{1}{n}\sum_{i=1}^n\left(\zeta_{i2}^2-1\right)\right| > \frac{t}{4}\right\} \le 4\exp\left\{-C_6 n\min(t,t^2)\right\}$$

for some absolute constant $C_6 > 0$. It suffices to bound $|S_\infty^{d-1}(1/2)|$. Since

$$\begin{aligned}
|S_\infty^{d-1}(1/2)| &= \mathcal{N}(1/2, \{\mathbf{v}\in\mathbb{R}^d : \|\mathbf{v}\|_\infty = 1\}, \|\cdot\|_\infty)\\
&\le \mathcal{N}(1/2, \{\mathbf{v}\in\mathbb{R}^d : \|\mathbf{v}\|_\infty \le 1\}, \|\cdot\|_\infty) \le 6^d,
\end{aligned}$$

it follows that

$$\begin{aligned}
&\mathbb{P}\left(\left\|\frac{1}{n}\sum_{i=1}^n\mathbf{x}_i\mathbf{x}_i^{\mathrm{T}} - \boldsymbol{\Sigma}\right\|_\infty > t\|\boldsymbol{\Sigma}\|_\infty\right)\\
&\quad\le \sum_{j=1}^d\sum_{\mathbf{v}\in S_\infty^{d-1}(1/2)}\mathbb{P}\left\{\frac{1}{n}\sum_{i=1}^n(\mathbf{e}_j^{\mathrm{T}}\mathbf{x}_i)(\mathbf{v}^{\mathrm{T}}\mathbf{x}_i) - \mathbf{e}_j^{\mathrm{T}}\boldsymbol{\Sigma}\mathbf{v} > \frac{t}{2}\|\boldsymbol{\Sigma}\|_\infty\right\}\\
&\quad\le 4d\exp\{d\log 6 - C_6 n\min(t,t^2)\}\\
&\quad\le 4\exp\{4d - C_6 n\min(t,t^2)\},
\end{aligned}$$

and the proof is thus completed. $\square$

5.2 Proofs for Chapter 3

5.2.1 A useful matrix decomposition

Before proceeding to the proofs of the main results, we present a slightly technical yet useful matrix decomposition that will be used throughout the proofs for Chapter 3.

Lemma 13 *Let $\mathbf{X}, \mathbf{X}_0 \in \mathbb{R}^{n\times d}$ be $n \times d$ matrices and $\mathbf{P}_0 = \mathbf{X}_0\mathbf{X}_0^{\mathrm{T}}$. Let $\mathbf{X} = \mathbf{U}\mathbf{S}^{1/2}\mathbf{V}^{\mathrm{T}}$ and $\mathbf{X}_0 = \mathbf{U}_0\mathbf{S}_0^{1/2}\mathbf{V}_0^{\mathrm{T}}$ be the singular value decomposition of $\mathbf{X}$ and $\mathbf{X}_0$, respectively, where $\mathbf{U}, \mathbf{U}_0 \in \mathbb{O}(n, d)$, $\mathbf{V}, \mathbf{V}_0 \in \mathbb{O}(d)$, and $\mathbf{S}^{1/2}, \mathbf{S}_0^{1/2}$ are diagonal matrices with non-negative entries. Further let $\mathbf{U}_0^{\mathrm{T}}\mathbf{U} = \mathbf{W}_1\mathbf{\Sigma}\mathbf{W}_2^{\mathrm{T}}$ be the singular value decomposition of $\mathbf{U}_0^{\mathrm{T}}\mathbf{U}$, where $\mathbf{W}_1, \mathbf{W}_2 \in \mathbb{O}(d)$, and $\mathbf{\Sigma}$ is the diagonal matrix of singular values of $\mathbf{U}_0^{\mathrm{T}}\mathbf{U}$. Denote $\mathbf{W}_{\mathbf{U}} = \mathbf{W}_1\mathbf{W}_2^{\mathrm{T}}$. Assume that $[\mathbf{U}, \mathbf{U}_\perp] \in \mathbb{O}(n)$, namely, the columns of $\mathbf{U}_\perp$ are orthonormal and spans the orthogonal complement of* $\mathrm{Span}(\mathbf{U})$*, and $\mathbf{P} = \mathbf{X}\mathbf{X}^{\mathrm{T}} + \mathbf{U}_\perp\mathbf{S}_\perp\mathbf{U}_\perp^{\mathrm{T}}$ for some diagonal $\mathbf{S}_\perp = \mathrm{diag}(\sigma_{d+1}, \ldots, \sigma_n)$, where $\sigma_1(\mathbf{X}) \geq \ldots \geq \sigma_d(\mathbf{X}) \geq \sigma_{d+1} \geq \ldots \geq \sigma_n$. Then the following decomposition holds:*

$$\begin{aligned}\mathbf{X}\mathbf{V} - \mathbf{X}_0\mathbf{V}_0\mathbf{W}_{\mathbf{U}} &= (\mathbf{P} - \mathbf{P}_0)\mathbf{U}_0\mathbf{S}_0^{-1/2}\mathbf{W}_{\mathbf{U}} + (\mathbf{P} - \mathbf{P}_0)\mathbf{U}_0(\mathbf{W}_{\mathbf{U}}\mathbf{S}^{-1/2} - \mathbf{S}_0^{-1/2}\mathbf{W}_{\mathbf{U}})\\ &\quad - \mathbf{U}_0\mathbf{U}_0^{\mathrm{T}}(\mathbf{P} - \mathbf{P}_0)\mathbf{U}_0\mathbf{W}_{\mathbf{U}}\mathbf{S}^{-1/2} + (\mathbf{I} - \mathbf{U}_0\mathbf{U}_0^{\mathrm{T}})(\mathbf{P} - \mathbf{P}_0)\mathbf{R}_3\mathbf{S}^{-1/2}\\ &\quad + \mathbf{R}_1\mathbf{S}^{1/2} + \mathbf{U}_0\mathbf{R}_2,\end{aligned}$$

where

$$\mathbf{R}_1 = \mathbf{U}_0\mathbf{U}_0^{\mathrm{T}}\mathbf{U} - \mathbf{U}_0\mathbf{W}_{\mathbf{U}}, \quad \mathbf{R}_2 = \mathbf{W}_{\mathbf{U}}\mathbf{S}^{1/2} - \mathbf{S}_0^{1/2}\mathbf{W}_{\mathbf{U}}, \quad \textit{and} \quad \mathbf{R}_3 = \mathbf{U} - \mathbf{U}_0\mathbf{W}_{\mathbf{U}}.$$

Proof. The proof is similar to that of Theorem 50 in [30], and we present it here for completeness. First write

$$\begin{aligned}\mathbf{X}\mathbf{V} - \mathbf{X}_0\mathbf{V}_0\mathbf{W}_{\mathbf{U}} &= \mathbf{U}\mathbf{S}^{1/2} - \mathbf{U}_0\mathbf{S}_0^{1/2}\mathbf{W}_{\mathbf{U}}\\ &= \mathbf{U}\mathbf{S}^{1/2} - \mathbf{U}_0\mathbf{W}_{\mathbf{U}}\mathbf{S}^{1/2} + \mathbf{U}_0\mathbf{W}_{\mathbf{U}}\mathbf{S}^{1/2} - \mathbf{U}_0\mathbf{S}_0^{1/2}\mathbf{W}_{\mathbf{U}}\\ &= (\mathbf{U} - \mathbf{U}_0\mathbf{U}_0^{\mathrm{T}}\mathbf{U})\mathbf{S}^{1/2} + (\mathbf{U}_0\mathbf{U}_0^{\mathrm{T}}\mathbf{U} - \mathbf{U}_0\mathbf{W}_{\mathbf{U}})\mathbf{S}^{1/2}\\ &\quad + \mathbf{U}_0(\mathbf{W}_{\mathbf{U}}\mathbf{S}^{1/2} - \mathbf{S}_0^{1/2}\mathbf{W}_{\mathbf{U}})\end{aligned}$$

$$= (\mathbf{U}\mathbf{S}^{1/2} - \mathbf{U}_0\mathbf{U}_0^{\mathrm{T}}\mathbf{U}\mathbf{S}^{1/2}) + \mathbf{R}_1\mathbf{S}^{1/2} + \mathbf{U}_0\mathbf{R}_2$$

$$= (\mathbf{P}\mathbf{U}\mathbf{S}^{-1/2} - \mathbf{U}_0\mathbf{U}_0^{\mathrm{T}}\mathbf{P}\mathbf{U}\mathbf{S}^{-1/2}) + \mathbf{R}_1\mathbf{S}^{1/2} + \mathbf{U}_0\mathbf{R}_2,$$

where the last equality is due to the fact that $\mathbf{P}\mathbf{U}\mathbf{S}^{-1/2} = \mathbf{U}\mathbf{S}^{1/2}$. Observe that $\mathbf{U}_0\mathbf{U}_0^{\mathrm{T}}\mathbf{P}_0 = \mathbf{P}_0$, then we re-arrange the term in the parenthesis in the preceding display to

$$\begin{aligned}
&\mathbf{P}\mathbf{U}\mathbf{S}^{-1/2} - \mathbf{U}_0\mathbf{U}_0^{\mathrm{T}}\mathbf{P}\mathbf{U}\mathbf{S}^{-1/2}\\
&\quad= (\mathbf{P} - \mathbf{P}_0)\mathbf{U}\mathbf{S}^{-1/2} - \mathbf{U}_0\mathbf{U}_0^{\mathrm{T}}(\mathbf{P} - \mathbf{P}_0)\mathbf{U}\mathbf{S}^{-1/2}\\
&\quad= (\mathbf{P} - \mathbf{P}_0)\mathbf{U}\mathbf{S}^{-1/2} - \mathbf{U}_0\mathbf{U}_0^{\mathrm{T}}(\mathbf{P} - \mathbf{P}_0)(\mathbf{U} - \mathbf{U}_0\mathbf{W}_{\mathbf{U}})\mathbf{S}^{-1/2}\\
&\quad- \mathbf{U}_0\mathbf{U}_0^{\mathrm{T}}(\mathbf{P} - \mathbf{P}_0)\mathbf{U}_0\mathbf{W}_{\mathbf{U}}\mathbf{S}^{-1/2}\\
&\quad= (\mathbf{P} - \mathbf{P}_0)\mathbf{U}\mathbf{S}^{-1/2} - \mathbf{U}_0\mathbf{U}_0^{\mathrm{T}}(\mathbf{P} - \mathbf{P}_0)\mathbf{R}_3\mathbf{S}^{-1/2}\\
&\quad- \mathbf{U}_0\mathbf{U}_0^{\mathrm{T}}(\mathbf{P} - \mathbf{P}_0)\mathbf{U}_0\mathbf{W}_{\mathbf{U}}\mathbf{S}^{-1/2}\\
&\quad= (\mathbf{P} - \mathbf{P}_0)\mathbf{U}\mathbf{S}^{-1/2} - (\mathbf{P} - \mathbf{P}_0)\mathbf{R}_3\mathbf{S}^{-1/2} + (\mathbf{I} - \mathbf{U}_0\mathbf{U}_0^{\mathrm{T}})(\mathbf{P} - \mathbf{P}_0)\mathbf{R}_3\mathbf{S}^{-1/2}\\
&\qquad- \mathbf{U}_0\mathbf{U}_0^{\mathrm{T}}(\mathbf{P} - \mathbf{P}_0)\mathbf{U}_0\mathbf{W}_{\mathbf{U}}\mathbf{S}^{-1/2}\\
&\quad= (\mathbf{P} - \mathbf{P}_0)\mathbf{U}_0\mathbf{W}_{\mathbf{U}}\mathbf{S}^{-1/2} + (\mathbf{I} - \mathbf{U}_0\mathbf{U}_0^{\mathrm{T}})(\mathbf{P} - \mathbf{P}_0)\mathbf{R}_3\mathbf{S}^{-1/2}\\
&\quad- \mathbf{U}_0\mathbf{U}_0^{\mathrm{T}}(\mathbf{P} - \mathbf{P}_0)\mathbf{U}_0\mathbf{W}_{\mathbf{U}}\mathbf{S}^{-1/2}.
\end{aligned}$$

We observe that

$$(\mathbf{P} - \mathbf{P}_0)\mathbf{U}_0\mathbf{W}_{\mathbf{U}}\mathbf{S}^{-1/2} = (\mathbf{P} - \mathbf{P}_0)\mathbf{U}_0\mathbf{S}_0^{-1/2}\mathbf{W}_{\mathbf{U}} + (\mathbf{P} - \mathbf{P}_0)\mathbf{U}_0(\mathbf{W}_{\mathbf{U}}\mathbf{S}^{-1/2} - \mathbf{S}_0^{-1/2}\mathbf{W}_{\mathbf{U}}),$$

and thus complete the proof. $\square$

When the embedding dimension d is one, we obtain immediately the following rank-one corollary:

Corollary 2 *Let $\mathbf{x}, \mathbf{x}_0 \in (0,1)^n$ be n-dimensional vectors. Denote $\mathbf{E} = \mathbf{x}\mathbf{x}^{\mathrm{T}} - \mathbf{x}_0\mathbf{x}_0^{\mathrm{T}}$. Then the following decomposition holds:*

$$\mathbf{x} - \mathbf{x}_0 = \frac{\mathbf{E}\mathbf{x}_0}{\|\mathbf{x}_0\|_2^2} + \left(\frac{1}{\|\mathbf{x}\|_2} - \frac{1}{\|\mathbf{x}_0\|_2}\right)\frac{\mathbf{E}\mathbf{x}_0}{\|\mathbf{x}_0\|_2} - \frac{\mathbf{x}_0\mathbf{x}_0^{\mathrm{T}}\mathbf{E}\mathbf{x}_0}{\|\mathbf{x}\|_2\|\mathbf{x}_0\|_2^3}$$

$$+\left(\mathbf{I}-\frac{\mathbf{x}_0\mathbf{x}_0^{\mathrm{T}}}{\|\mathbf{x}_0\|_2^2}\right)\frac{\mathbf{E}}{\|\mathbf{x}\|_2}\left(\frac{\mathbf{x}}{\|\mathbf{x}\|_2}-\frac{\mathbf{x}_0}{\|\mathbf{x}_0\|_2}\right)$$
$$+\|\mathbf{x}\|\left(\frac{\mathbf{x}_0\mathbf{x}_0^{\mathrm{T}}\mathbf{x}}{\|\mathbf{x}_0\|_2^2\|\mathbf{x}\|_2}-\frac{\mathbf{x}_0}{\|\mathbf{x}_0\|_2}\right)+(\|\mathbf{x}\|_2-\|\mathbf{x}_0\|_2)\frac{\mathbf{x}_0}{\|\mathbf{x}_0\|_2}.$$

Furthermore, the following inequality holds:

$$\|\mathbf{x}-\mathbf{x}_0\|_2\le\frac{3\|\mathbf{E}\|_2}{\|\mathbf{x}_0\|}+\frac{4\|\mathbf{E}\|_2}{\|\mathbf{x}\|_2}+4\frac{(\|\mathbf{x}\|_2^2+\|\mathbf{x}_0\|_2^2)\|\mathbf{E}\|_2}{\|\mathbf{x}_0\|_2^2\|\mathbf{x}\|_2}.$$

Proof. We first prove the decomposition result. It suffices to show that $\mathbf{W}_{\mathbf{U}}=1$. In fact, $\mathbf{x}^{\mathrm{T}}\mathbf{x}_0>0$, it follows that $(\mathbf{x}^{\mathrm{T}}\mathbf{x}_0)/(\|\mathbf{x}\|_2\|\mathbf{x}_0\|_2)\in(0,1)$. Therefore, we can choose the left and right singular vectors of $(\mathbf{x}^{\mathrm{T}}\mathbf{x}_0)/(\|\mathbf{x}\|_2\|\mathbf{x}_0\|_2)$ to be 1, and consequently, the corresponding orthogonal matrix $\mathbf{W}_{\mathbf{U}}\in\mathbb{R}^{1\times1}$ is also 1.

Now we move forward to prove the inequality result. Observe that

$$\left\|\left(\frac{1}{\|\mathbf{x}\|_2}-\frac{1}{\|\mathbf{x}_0\|_2}\right)\frac{\mathbf{E}\mathbf{x}_0}{\|\mathbf{x}_0\|_2}-\frac{\mathbf{x}_0\mathbf{x}_0^{\mathrm{T}}\mathbf{E}\mathbf{x}_0}{\|\mathbf{x}\|_2\|\mathbf{x}_0\|_2^3}+\left(\mathbf{I}-\frac{\mathbf{x}_0\mathbf{x}_0^{\mathrm{T}}}{\|\mathbf{x}_0\|_2^2}\right)\frac{\mathbf{E}}{\|\mathbf{x}\|_2}\left(\frac{\mathbf{x}}{\|\mathbf{x}\|_2}-\frac{\mathbf{x}_0}{\|\mathbf{x}_0\|_2}\right)\right\|_2$$
$$\le\left(\frac{1}{\|\mathbf{x}\|_2}+\frac{1}{\|\mathbf{x}_0\|_2}\right)\|\mathbf{E}\|_2+\frac{\|\mathbf{E}\|_2}{\|\mathbf{x}\|_2}+\frac{2\|\mathbf{E}\|_2}{\|\mathbf{x}\|_2}=\frac{\|\mathbf{E}\|_2}{\|\mathbf{x}_0\|_2}+\frac{4\|\mathbf{E}\|_2}{\|\mathbf{x}\|_2}.$$

In addition, by Davis-Kahan theorem and Weyl's inequality,

$$\left\|\frac{\mathbf{x}_0\mathbf{x}_0^{\mathrm{T}}\mathbf{x}}{\|\mathbf{x}_0\|_2^2\|\mathbf{x}\|_2}-\frac{\mathbf{x}_0}{\|\mathbf{x}_0\|_2}\right\|_2\le\frac{4\|\mathbf{E}\|_2^2}{\|\mathbf{x}\|_2^2\|\mathbf{x}_0\|_2^2}\le\frac{4(\|\mathbf{x}\|_2^2+\|\mathbf{x}_0\|_2^2)\|\mathbf{E}\|_2}{\|\mathbf{x}\|_2^2\|\mathbf{x}_0\|_2^2},$$
$$|\|\mathbf{x}\|_2-\|\mathbf{x}_0\|_2|=\frac{|\|\mathbf{x}\|_2^2-\|\mathbf{x}_0\|_2^2|}{\|\mathbf{x}\|_2+\|\mathbf{x}_0\|_2}\le\frac{\|\mathbf{E}\|_2}{\|\mathbf{x}_0\|}.$$

The proof is then completed by combining above derivations. □

5.2.2 Proof of the minimax lower bound

It is routine to leverage Fano's lemma and its variations to derive minimax lower bounds for a wide class of statistical problems. Specifically, we will rely on the following version of Fano's lemma to construct the minimax lower bound for the random dot product graph model. For a totally bounded pseudo-metric space (T,ρ), for any $\epsilon>0$, the covering number $\mathcal{N}(\epsilon,T,\rho)$ is the minimum number of balls of radius ϵ (with respect to the metric ρ) that are needed to cover T.

Lemma 14 (Proposition 3, [28]) *Let (Θ, ρ) be a totally bounded pseudo-metric space and $\{\mathbb{P}_\theta : \theta \in \Theta\}$ a collection of distributions. Let $A = \sup_{\theta \neq \theta'} D(\mathbb{P}_\theta || \mathbb{P}_{\theta'})/\rho^2(\theta, \theta')$. If there exist $0 < c_0 < c_1 < \infty$, $\epsilon_0 > 0$, and $\alpha \geq 1$ such that*

$$\left(\frac{c_0}{\epsilon}\right)^\alpha \leq \mathcal{N}(\epsilon, \Theta, \rho) \leq \left(\frac{c_1}{\epsilon}\right)^\alpha$$

for all $\epsilon \in (0, \epsilon_0)$, then

$$\inf_{\hat{\theta}} \sup_{\theta \in \Theta} \mathbb{E}_\theta\{\rho^2(\hat{\theta}, \theta)\} \geq \frac{c_0^2}{840 c_1^2} \min\left(\frac{\alpha}{A}, \epsilon_0^2\right).$$

We now proceed to the proof of Theorem 6.

Proof. Consider the following subset of latent positions:

$$\widetilde{\Theta}_n = \left\{\mathbf{X} = [\mathbf{x}_1, \ldots, \mathbf{x}_n]^{\mathrm{T}} \in \mathcal{X}^n : \sqrt{\frac{1}{4d}} \leq x_{i1} \leq \sqrt{\frac{3}{4d}}, x_{i2} = \ldots = x_{id} = \sqrt{\frac{1}{4d}}, i \in [n]\right\}.$$

Let $\mathbf{X}_1, \mathbf{X}_2 \in \widetilde{\Theta}_n$, and let $\mathbf{u}_1, \mathbf{u}_2$ be the first columns of $\mathbf{X}_1$ and $\mathbf{X}_2$, respectively. Clearly,

$$\|\mathbf{X}_1\mathbf{X}_1^{\mathrm{T}} - \mathbf{X}_2\mathbf{X}_2^{\mathrm{T}}\|_{\mathrm{F}}^2 = \sum_{i=1}^n \sum_{j=1}^n (x_{1i1}x_{1j1} - x_{2i1}x_{2j1})^2 = \|\mathbf{u}_1\mathbf{u}_1^{\mathrm{T}} - \mathbf{u}_2\mathbf{u}_2^{\mathrm{T}}\|_{\mathrm{F}}^2.$$

Since $\mathbf{X}_1, \mathbf{X}_2 \in \widetilde{\Theta}_n$, it follows that $\sqrt{n/(4d)} \leq \|\mathbf{u}_1\|_2, \|\mathbf{u}_2\|_2 \leq \sqrt{3n/(4d)}$. Applying Corollary 2 yields

$$\begin{aligned}\|\mathbf{u}_1 - \mathbf{u}_2\|_2 &\leq (14\sqrt{d})\frac{\|\mathbf{X}_1\mathbf{X}_1^{\mathrm{T}} - \mathbf{X}_2\mathbf{X}_2^{\mathrm{T}}\|_2}{\sqrt{n}} + \frac{(6n/d)\|\mathbf{X}_1\mathbf{X}_1^{\mathrm{T}} - \mathbf{X}_2\mathbf{X}_2^{\mathrm{T}}\|_2}{(n/4d)^{3/2}} \\ &\leq \frac{62\sqrt{d}\|\mathbf{X}_1\mathbf{X}_1^{\mathrm{T}} - \mathbf{X}_2\mathbf{X}_2^{\mathrm{T}}\|_2}{\sqrt{n}}.\end{aligned}$$

Let $\rho : \widetilde{\Theta}_n \times \widetilde{\Theta}_n \to [0, \infty)$ be defined by $\rho(\mathbf{X}_1, \mathbf{X}_2) = (1/n)\|\mathbf{X}_1\mathbf{X}_1^{\mathrm{T}} - \mathbf{X}_2\mathbf{X}_2^{\mathrm{T}}\|_{\mathrm{F}}$. It follows that there exists some $\epsilon_0 > 0$, such that for all $\epsilon \in (0, \epsilon_0)$,

$$\begin{aligned}\mathcal{N}(\epsilon, \widetilde{\Theta}_n, \rho) &\geq \mathcal{N}\left\{\epsilon, \left(\sqrt{\frac{1}{4d}}, \sqrt{\frac{3}{4d}}\right)^n, \frac{1}{62\sqrt{nd}}\|\cdot\|_2\right\} \\ &= \mathcal{N}\left\{62\sqrt{nd}\epsilon, \left(\sqrt{\frac{1}{4d}}, \sqrt{\frac{3}{4d}}\right)^n, \|\cdot\|_2\right\}.\end{aligned}$$

For any $\Theta \subset \mathbb{R}^n$, a standard volume comparison argument yields

$$\left(\frac{1}{\epsilon}\right)^n \frac{\mathrm{vol}(\Theta)}{\mathrm{vol}(B_1^n)} \leq \mathcal{N}(\epsilon, \Theta, \|\cdot\|_2).$$

Hence,

$$\mathcal{N}\left\{62\sqrt{nd}\epsilon, \left(\sqrt{\frac{1}{4d}}, \sqrt{\frac{3}{4d}}\right)^n, \|\cdot\|_2\right\} \geq \left(\frac{1}{62\sqrt{nd}\epsilon}\right)^n \left(\frac{\sqrt{3}-1}{2\sqrt{d}}\right)^n \frac{\Gamma(n/2+1)}{\pi^{n/2}}.$$

Observe that by Stirling's formula, for sufficiently large n, it holds that

$$\left(\frac{n}{2\mathrm{e}}\right)^{n/2} \geq \frac{\Gamma\left(n/2+1\right)}{\pi^{n/2}} \geq \left(\frac{n}{2\pi\mathrm{e}}\right)^{n/2}. \tag{5.13}$$

Thus, we obtain the following lower bound for the covering number of $\widetilde{\Theta}_n$:

$$\mathcal{N}(\epsilon, \Theta_n, \rho) \geq \left\{\frac{(\sqrt{3}-1)}{124\sqrt{2\pi\mathrm{e}}\sqrt{d}\epsilon}\right\}^n.$$

We proceed to derive an upper bound for the covering number. Note that

$$\begin{aligned}\|\mathbf{X}_1\mathbf{X}_1^{\mathrm{T}} - \mathbf{X}_2\mathbf{X}_2^{\mathrm{T}}\|_{\mathrm{F}} = \|\mathbf{u}_1\mathbf{u}_1^{\mathrm{T}} - \mathbf{u}_2\mathbf{u}_2^{\mathrm{T}}\|_{\mathrm{F}} &\leq \|\mathbf{u}_1(\mathbf{u}_1-\mathbf{u}_2)^{\mathrm{T}}\|_{\mathrm{F}} + \|(\mathbf{u}_1-\mathbf{u}_2)\mathbf{u}_2^{\mathrm{T}}\|_{\mathrm{F}} \\ &\leq \sqrt{\frac{3n}{d}}\|\mathbf{u}_1-\mathbf{u}_2\|_2.\end{aligned}$$

This further implies that

$$\mathcal{N}(\epsilon, \widetilde{\Theta}_n, \rho) \leq \mathcal{N}\left(\sqrt{\frac{nd}{3}}\epsilon, \left[\sqrt{\frac{1}{4d}}, \sqrt{\frac{3}{4d}}\right]^n, \|\cdot\|_2\right) \leq \left\{\frac{4\sqrt{3}(\sqrt{3}-1)}{\sqrt{2\mathrm{e}}\sqrt{d}\epsilon}\right\}^n$$

by a simple volume comparison argument. Hence, we obtain the following estimate of the covering number:

$$\left(\frac{c_0}{\sqrt{d}\epsilon}\right)^n \leq \mathcal{N}(\epsilon, \widetilde{\Theta}_n, \rho) \leq \left(\frac{c_1}{\sqrt{d}\epsilon}\right)^n \tag{5.14}$$

for some constants $0 < c_0 < c_1 < \infty$. It remains to derive

$$A = \sup_{\mathbf{X}_1, \mathbf{X}_2 \in \widetilde{\Theta}_n} \{D(\mathbb{P}_{\mathbf{X}_1}||\mathbb{P}_{\mathbf{X}_2})/\rho^2(\mathbf{X}_1, \mathbf{X}_2)\}.$$

For any $\mathbf{X}_1, \mathbf{X}_2 \in \widetilde{\Theta}_n$, write

$$\frac{D(\mathbb{P}_{\mathbf{X}_1}||\mathbb{P}_{\mathbf{X}_2})}{\rho^2(\mathbf{X}_1, \mathbf{X}_2)} = \frac{n^2}{\|\mathbf{X}_1\mathbf{X}_1^{\mathrm{T}} - \mathbf{X}_2\mathbf{X}_2^{\mathrm{T}}\|_{\mathrm{F}}^2}$$

$$
\begin{aligned}
&\times \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left\{(u_{1i}u_{1j})\log\left(\frac{u_{1i}u_{1j}}{u_{2i}u_{2j}}\right)+(1-u_{1i}u_{1j})\log\left(\frac{1-u_{1i}u_{1j}}{1-u_{2i}u_{2j}}\right)\right\}\\
&\leq \frac{n^2}{\|\mathbf{X}_1\mathbf{X}_1^{\mathrm{T}}-\mathbf{X}_2\mathbf{X}_2^{\mathrm{T}}\|_{\mathrm{F}}^2}\\
&\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left\{(u_{1i}u_{1j})\left(\frac{u_{1i}u_{1j}}{u_{2i}u_{2j}}-1\right)+(1-u_{1i}u_{1j})\left(\frac{1-u_{1i}u_{1j}}{1-u_{2i}u_{2j}}-1\right)\right\}\\
&\leq \frac{n^2}{\|\mathbf{X}_1\mathbf{X}_1^{\mathrm{T}}-\mathbf{X}_2\mathbf{X}_2^{\mathrm{T}}\|_{\mathrm{F}}^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{(u_{1i}u_{1j}-u_{2i}u_{2j})^2}{u_{2i}u_{2j}(1-u_{2i}u_{2j})}\\
&\leq \frac{16d^2n^2}{\|\mathbf{X}_1\mathbf{X}_1^{\mathrm{T}}-\mathbf{X}_2\mathbf{X}_2^{\mathrm{T}}\|_{\mathrm{F}}^2}\sum_{i=1}^{n}\sum_{j=1}^{n}(u_{1i}u_{1j}-u_{2i}u_{2j})^2=16d^2n^2.
\end{aligned}
$$

Hence $A\leq 16d^2n^2$. Applying Lemma 14 and the covering number estimate (5.14) yields that

$$
\inf_{\widehat{\mathbf{X}}}\sup_{\mathbf{X}\in\widetilde{\Theta}_n}\mathbb{E}_{\mathbf{X}}\left(\frac{1}{n^2}\|\widehat{\mathbf{X}}\widehat{\mathbf{X}}^{\mathrm{T}}-\mathbf{X}\mathbf{X}^{\mathrm{T}}\|_{\mathrm{F}}^2\right)\gtrsim\frac{1}{n}.
$$

Finally, observe that for any $\mathbf{W}\in\mathbb{O}(d)$,

$$
\begin{aligned}
\|\widehat{\mathbf{X}}\widehat{\mathbf{X}}^{\mathrm{T}}-\mathbf{X}\mathbf{X}^{\mathrm{T}}\|_{\mathrm{F}}&\leq\|(\widehat{\mathbf{X}}-\mathbf{X}\mathbf{W})\widehat{\mathbf{X}}^{\mathrm{T}}\|_{\mathrm{F}}+\|(\mathbf{X}\mathbf{W})(\widehat{\mathbf{X}}-\mathbf{X}\mathbf{W})^{\mathrm{T}}\|_{\mathrm{F}}\\
&\leq\|\widehat{\mathbf{X}}-\mathbf{X}\mathbf{W}\|_{\mathrm{F}}(\|\widehat{\mathbf{X}}\|_{\mathrm{F}}+\|\mathbf{X}\|_{\mathrm{F}})\lesssim\sqrt{n}\|\widehat{\mathbf{X}}-\mathbf{X}\mathbf{W}\|_{\mathrm{F}}
\end{aligned}
$$

by the assumption that $\|\widehat{\mathbf{X}}\|_{\mathrm{F}}\lesssim\sqrt{n}$ with probability one. Namely,

$$
\inf_{\mathbf{W}\in\mathbb{O}(d)}\|\widehat{\mathbf{X}}-\mathbf{X}\mathbf{W}\|_{\mathrm{F}}\gtrsim(1/\sqrt{n})\|\widehat{\mathbf{X}}\widehat{\mathbf{X}}^{\mathrm{T}}-\mathbf{X}\mathbf{X}^{\mathrm{T}}\|_{\mathrm{F}},
$$

and this implies that

$$
\inf_{\widehat{\mathbf{X}}}\sup_{\mathbf{X}\in\widetilde{\Theta}_n}\mathbb{E}_{\mathbf{X}}\left(\frac{1}{n}\inf_{\mathbf{W}\in\mathbb{O}(d)}\|\widehat{\mathbf{X}}-\mathbf{X}\mathbf{W}\|_{\mathrm{F}}^2\right)\gtrsim\inf_{\widehat{\mathbf{X}}}\sup_{\mathbf{X}\in\widetilde{\Theta}_n}\mathbb{E}_{\mathbf{X}}\left(\frac{1}{n^2}\|\widehat{\mathbf{X}}\widehat{\mathbf{X}}^{\mathrm{T}}-\mathbf{X}\mathbf{X}^{\mathrm{T}}\|_{\mathrm{F}}^2\right)\gtrsim\frac{1}{n},
$$

completing the proof. □

5.2.3 Proofs for Section 3.2

The blueprint of the proof of Theorem 7 can be described as a "prior-mass-and-testing" technique originally presented in the seminal work [51]. Roughly speaking, the "prior-mass" technique is to show the denominator

$$
D_n=\int_{\mathcal{X}^n}\prod_{i\leq j}\frac{p(A_{ij}\mid\mathbf{X})}{p(A_{ij}\mid\mathbf{X}_0)}\Pi(\mathrm{d}\mathbf{X})
$$

appearing in the posterior distribution can be bounded from below with large probability, and the "testing" technique requires the construction of suitable test functions. In what follows we formalize these steps.

Bounding the denominator from below

Lemma 15 *Let* $\mathbf{A} \sim \mathrm{RDPG}(\mathbf{X}_0)$ *for some* $\mathbf{X}_0 \in \mathcal{X}^n$*. Assume that* $\delta \leq \min_{i,j} \mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j} \leq \max_{i,j} \mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j} \leq 1-\delta$ *for some constant* $\delta \in (0, 1/2)$ *independent of* n*, and that* $\pi_{\mathbf{x}}$ *is bounded away from* 0 *and* ∞*. Then for any constants* $\beta, \gamma > 0$*, and for sufficiently large* n *and sufficiently small* $\epsilon > 0$*,*

$$\mathbb{P}_0\left[D_n \leq \exp\left\{(c_\pi + d\log\beta)n - \left(\frac{16\beta^2}{\delta^2}+\gamma\right)n^2\epsilon^2 - nd\left(\log\frac{1}{\epsilon}\right)\right\}\right] \leq \exp\left(-\frac{\gamma^2\delta^4 n^2\epsilon^2}{128\beta^2}\right)$$

for some constant c_π *independent of* n *and* d*.*

Proof. For any constant $\beta > 0$, set $\mathcal{E}_n = \{\mathbf{X} : \|\mathbf{X}-\mathbf{X}_0\|_{2\to\infty} < \beta\epsilon\}$. Denote $\mathbf{P} = \mathbf{X}\mathbf{X}^{\mathrm{T}} = [\mathbf{P}_{ij}]_{n\times n}$ and $\mathbf{P}_0 = \mathbf{X}_0\mathbf{X}_0^{\mathrm{T}} = [\mathbf{P}_{0ij}]_{n\times n}$. Write

$$\begin{aligned}
D_n &\geq \Pi(\mathcal{E}_n)\int_{\mathcal{E}_n}\prod_{i\leq j}\frac{p(A_{ij}\mid\mathbf{X})}{p(A_{ij}\mid\mathbf{X}_0)}\Pi(\mathrm{d}\mathbf{X}\mid\mathcal{E}_n)\\
&= \Pi(\mathcal{E}_n)\int_{\mathcal{E}_n}\exp\left[\sum_{i=1}^n\sum_{j=i}^n\left\{-A_{ij}\log\left(\frac{\mathbf{P}_{0ij}}{\mathbf{P}_{ij}}\right)-(1-A_{ij})\log\left(\frac{1-\mathbf{P}_{0ij}}{1-\mathbf{P}_{ij}}\right)\right\}\right]\Pi(\mathrm{d}\mathbf{X}\mid\mathcal{E}_n)\\
&\geq \Pi(\mathcal{E}_n)\int_{\mathcal{E}_n}\exp\left[\sum_{i=1}^n\sum_{j=i}^n\left\{-A_{ij}\left(\frac{\mathbf{P}_{0ij}-\mathbf{P}_{ij}}{\mathbf{P}_{ij}}\right)-(1-A_{ij})\left(\frac{\mathbf{P}_{ij}-\mathbf{P}_{0ij}}{1-\mathbf{P}_{ij}}\right)\right\}\right]\Pi(\mathrm{d}\mathbf{X}\mid\mathcal{E}_n)\\
&= \Pi(\mathcal{E}_n)\int_{\mathcal{E}_n}\exp\left[-\sum_{i=1}^n\sum_{j=i}^n\left\{(A_{ij}-\mathbf{P}_{0ij})\frac{(\mathbf{P}_{0ij}-\mathbf{P}_{ij})}{\mathbf{P}_{ij}(1-\mathbf{P}_{ij})}+\frac{(\mathbf{P}_{0ij}-\mathbf{P}_{ij})^2}{\mathbf{P}_{ij}(1-\mathbf{P}_{ij})}\right\}\right]\Pi(\mathrm{d}\mathbf{X}\mid\mathcal{E}_n)\\
&\geq \Pi(\mathcal{E}_n)\exp\left\{-\sum_{i=1}^n\sum_{j=i}^n(A_{ij}-\mathbf{P}_{0ij})\int_{\mathcal{E}_n}\frac{(\mathbf{P}_{ij}-\mathbf{P}_{0ij})}{\mathbf{P}_{ij}(1-\mathbf{P}_{ij})}\Pi(\mathrm{d}\mathbf{X}\mid\mathcal{E}_n)\right\}\\
&\quad\times\exp\left\{-\sum_{i=1}^n\sum_{j=i}^n\int_{\mathcal{E}_n}\frac{(\mathbf{P}_{0ij}-\mathbf{P}_{ij})^2}{\mathbf{P}_{ij}(1-\mathbf{P}_{ij})}\Pi(\mathrm{d}\mathbf{X}\mid\mathcal{E}_n)\right\},
\end{aligned}$$

where the third line follows from the fact that $\log x \leq (x-1)$ for all $x > 0$, and the last inequality is due to Jensen's inequality. Since for any $\mathbf{X} \in \mathcal{E}_n$, we have, for any

$i, j \in [n]$,

$$
\begin{aligned}
|\mathbf{P}_{ij} - \mathbf{P}_{0ij}| &\leq (\|\mathbf{x}_j\|_2 + \|\mathbf{x}_{0i}\|_2)\|\mathbf{X} - \mathbf{X}_0\|_{2\to\infty} \leq \frac{\delta}{2},\\
\mathbf{P}_{ij}(1-\mathbf{P}_{ij}) &\geq (\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j} - |\mathbf{P}_{ij} - \mathbf{P}_{0ij}|)(1 - \mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j} - |\mathbf{P}_{ij} - \mathbf{P}_{0ij}|) \geq \frac{\delta^2}{4},\\
\left|\frac{\mathbf{P}_{ij} - \mathbf{P}_{0ij}}{\mathbf{P}_{ij}(1-\mathbf{P}_{ij})}\right| &\leq \max_{i,j\in[n]} \frac{|(\mathbf{x}_i - \mathbf{x}_{0i})^{\mathrm{T}}\mathbf{x}_j| + |\mathbf{x}_{0i}^{\mathrm{T}}(\mathbf{x}_j - \mathbf{x}_{0j})|}{\mathbf{P}_{ij}(1-\mathbf{P}_{ij})} \leq \frac{2\|\mathbf{X} - \mathbf{X}_0\|_{2\to\infty}}{\delta^2/4} \leq \frac{8\beta\epsilon}{\delta^2},\\
\frac{(\mathbf{P}_{ij} - \mathbf{P}_{0ij})^2}{\mathbf{P}_{ij}(1-\mathbf{P}_{ij})} &\leq \max_{i,j\in[n]} \frac{2|(\mathbf{x}_i - \mathbf{x}_{0i})^{\mathrm{T}}\mathbf{x}_j|^2 + 2|\mathbf{x}_{0i}^{\mathrm{T}}(\mathbf{x}_j - \mathbf{x}_{0j})|^2}{\mathbf{P}_{ij}(1-\mathbf{P}_{ij})} \leq \frac{16\beta^2\epsilon^2}{\delta^2},
\end{aligned}
$$

implying that

$$
\left|\int_{\mathcal{E}_n} \frac{(\mathbf{P}_{ij} - \mathbf{P}_{0ij})}{\mathbf{P}_{ij}(1-\mathbf{P}_{ij})}\Pi(\mathrm{d}\mathbf{X} \mid \mathcal{E}_n)\right| \leq \frac{8\beta\epsilon}{\delta^2}, \quad \int_{\mathcal{E}_n} \frac{(\mathbf{P}_{ij} - \mathbf{P}_{0ij})^2}{\mathbf{P}_{ij}(1-\mathbf{P}_{ij})}\Pi(\mathrm{d}\mathbf{X} \mid \mathcal{E}_n) \leq \frac{16\beta^2\epsilon^2}{\delta^2},
$$

it follows that for any $\gamma > 0$,

$$
\begin{aligned}
&D_n \leq \Pi(\mathcal{E}_n)\exp\left\{-\frac{n(n+1)}{2}\left(\frac{16\beta^2\epsilon^2}{\delta^2}\right) - \gamma n^2\epsilon^2\right\}\\
&\quad\Longrightarrow \sum_{i=1}^n\sum_{j=i}^n (A_{ij} - \mathbf{P}_{0ij})\int_{\mathcal{E}_n} \frac{(\mathbf{P}_{ij} - \mathbf{P}_{0ij})}{\mathbf{P}_{0ij}(1-\mathbf{P}_{0ij})}\Pi(\mathrm{d}\mathbf{X} \mid \mathcal{E}_n) > \gamma n^2\epsilon^2.
\end{aligned}
$$

Hence, by Hoeffding's inequality,

$$
\begin{aligned}
&\mathbb{P}_0\left[D_n \leq \Pi(\mathcal{E}_n)\exp\left\{-\frac{n(n+1)}{2}\left(\frac{16\beta^2\epsilon^2}{\delta^2}\right) - \gamma n^2\epsilon^2\right\}\right]\\
&\quad\leq \mathbb{P}_0\left\{\sum_{i=1}^n\sum_{j=i}^n (A_{ij} - \mathbf{P}_{0ij})\int_{\mathcal{E}_n} \frac{(\mathbf{P}_{ij} - \mathbf{P}_{0ij})}{\mathbf{P}_{0ij}(1-\mathbf{P}_{0ij})}\Pi(\mathrm{d}\mathbf{X} \mid \mathcal{E}_n) > \gamma n^2\epsilon^2\right\}\\
&\quad\leq \exp\left\{-\frac{4\gamma^2 n^4\epsilon^4}{n(n+1)}\left(\frac{\delta^2}{16\beta\epsilon}\right)^2\right\} = \exp\left(-\frac{\gamma^2\delta^4 n^2\epsilon^2}{128\beta^2}\right).
\end{aligned}
$$

It suffices to provide an exponential lower bound for $\Pi(\mathcal{E}_n)$. This can be easily obtained using the fact that $\pi_{\mathbf{x}}(\mathbf{x}_i)\mathrm{vol}(B_1^d) \geq \exp(c_\pi) > 0$ for some constant c_π: for sufficiently small ϵ, $\{\|\mathbf{x}_i - \mathbf{x}_{0i}\|_2 < \beta\epsilon\} \subset \mathcal{X}$, and thus,

$$
\begin{aligned}
\Pi(\mathcal{E}_n) &= \prod_{i=1}^n \int_{\{\|\mathbf{x}_i - \mathbf{x}_{0i}\|_2 < \beta\epsilon\}} \pi_{\mathbf{x}}(\mathbf{x}_i)\mathrm{d}\mathbf{x}_i \geq \prod_{i=1}^n \left\{\exp(c_\pi)\frac{\mathrm{vol}\left(\mathbf{x} : \|\mathbf{x} - \mathbf{x}_{0i}\|_2 < \beta\epsilon\right)}{\mathrm{vol}(B_1^d)}\right\}\\
&= \exp\left(nc_\pi\right)(\beta\epsilon)^{nd} = \exp\left\{n(c_\pi + d\log\beta) - nd\left(\log\frac{1}{\epsilon}\right)\right\}.
\end{aligned}
\tag{5.15}
$$

Namely, for any $\beta, \gamma > 0$, we obtain the following conclusion,

$$
\mathbb{P}_0\left[D_n \leq \exp\left\{(c_\pi + d\log\beta)\, n - \left(\frac{16\beta^2}{\delta^2} + \gamma\right)n^2\epsilon^2 - nd\left(\log\frac{1}{\epsilon}\right)\right\}\right] \leq \exp\left(-\frac{\gamma^2\delta^4 n^2\epsilon^2}{128\beta^2}\right),
$$

where c_π is some constant depending independent of d and n. The proof is thus completed. $\square$

Construction of test functions

Lemma 16 *Let $M > 0$, consider the pseudo-metric $\rho(\mathbf{X}, \mathbf{X}_0) = \|\mathbf{X}\mathbf{X}^{\mathrm{T}} - \mathbf{X}_0\mathbf{X}_0^{\mathrm{T}}\|_{\mathrm{F}}/n$, and take $\Theta_n = \left\{\mathbf{X} \in \mathcal{X}^n : \rho(\mathbf{X}, \mathbf{X}_0) \leq M\sqrt{(d\log n)n}\right\}$. Assume that $\sigma_d(\mathbf{X}_0) \geq \sigma_0\sqrt{n/d}$ for some constant $\sigma_0 > 0$ that is independent of n and d. If $(d^4 \log n)/n \to 0$ and $\pi_{\mathbf{x}}$ is bounded away from 0 and ∞, then the following inequalities hold for sufficiently large n:*

$$\mathcal{N}\left[\frac{\epsilon}{4}, \{\mathbf{X} \in \Theta_n : \rho(\mathbf{X}, \mathbf{X}_0) < \epsilon\}, \rho\right] \leq \left(\frac{3}{\epsilon}\right)^{d^2} \left\{24\sqrt{d}\left(1 + \frac{16}{\sigma_0} + \frac{8}{\sigma_0^2}\right)\right\}^{nd}$$

for all $\epsilon > 0$, and

$$\Pi\{\mathbf{X} \in \Theta_n : \rho(\mathbf{X}, \mathbf{X}_0) \leq 2j\epsilon\} \leq \left(\frac{3}{\epsilon}\right)^{d^2} \exp\left[\{C_\pi - \log\mathrm{vol}(B_1^d)\}n\right]\left(\sqrt{2\pi\mathrm{e}}Cj\epsilon\right)^{nd}$$

for all sufficiently small $\epsilon > 0$.

Proof. Denote $\mathcal{F} = \{\mathbf{X} \in \Theta_n : \rho(\mathbf{X}, \mathbf{X}_0) < \epsilon\}$. We first show that for any $\mathbf{X} \in \Theta_n$, $\sigma_d(\mathbf{X}) \geq \sigma_0\sqrt{n/d}/2$ for sufficiently large n. For any $\mathbf{X} \in \mathcal{F}$, by the Weyl's inequality, we have, for sufficiently large n,

$$\begin{aligned}
|\sigma_d(\mathbf{X}) - \sigma_d(\mathbf{X}_0)| &= \frac{|\lambda_d(\mathbf{X}\mathbf{X}^{\mathrm{T}}) - \lambda_d(\mathbf{X}_0\mathbf{X}_0^{\mathrm{T}})|}{\sigma_d(\mathbf{X}) + \sigma_d(\mathbf{X}_0)} \leq \sqrt{\frac{d}{\sigma_0^2 n}}\|\mathbf{X}\mathbf{X}^{\mathrm{T}} - \mathbf{X}_0\mathbf{X}_0^{\mathrm{T}}\|_{\mathrm{F}} \\
&= \sqrt{\frac{nd}{\sigma_0^2}}\rho(\mathbf{X}, \mathbf{X}_0) \leq \frac{\sigma_0}{2}\sqrt{\frac{n}{d}},
\end{aligned}$$

and hence, $\sigma_d(\mathbf{X}) \geq \sigma_d(\mathbf{X}_0) - |\sigma_d(\mathbf{X}) - \sigma_d(\mathbf{X}_0)| \geq \sigma_0\sqrt{n/d}/2$.

Let $\mathbf{X}_1, \mathbf{X}_2 \in \Theta_n$, and let them yield singular value decompositions $\mathbf{X}_1 = \mathbf{U}_1\mathbf{S}_1^{1/2}\mathbf{V}_1^{\mathrm{T}}$ and $\mathbf{X}_2 = \mathbf{U}_2\mathbf{S}_2^{1/2}\mathbf{V}_2^{\mathrm{T}}$, where $\mathbf{U}_1, \mathbf{U}_2 \in \mathbb{O}(n, d)$ and $\mathbf{V}_1, \mathbf{V}_2 \in \mathbb{O}(d)$. Further let $\mathbf{U}_2^{\mathrm{T}}\mathbf{U}_1 = \mathbf{W}_1\mathbf{\Sigma}\mathbf{W}_2^{\mathrm{T}}$ be the singular value decomposition of $\mathbf{U}_2^{\mathrm{T}}\mathbf{U}_1$, and let $\mathbf{W}_{\mathbf{U}} =$

$\mathbf{W}_1\mathbf{W}_2^{\mathrm{T}}$. Denote $\mathbf{P}_1 = \mathbf{X}_1\mathbf{X}_1^{\mathrm{T}}$ and $\mathbf{P}_2 = \mathbf{X}_2\mathbf{X}_2^{\mathrm{T}}$. Then by Lemma 13 and the fact that $\|\mathbf{A}\mathbf{B}\|_{\mathrm{F}} \leq \|\mathbf{A}\|_{\mathrm{F}}\|\mathbf{B}\|_2$, we have,

$$\begin{aligned}
\|\mathbf{X}_1\mathbf{V}_1 - \mathbf{X}_2\mathbf{V}_2\mathbf{W}_{\mathbf{U}}\|_{\mathrm{F}} \leq{}& \|\mathbf{S}_2^{-1/2}\|_2\|\mathbf{P}_1 - \mathbf{P}_2\|_{\mathrm{F}} + \|\mathbf{P}_1 - \mathbf{P}_2\|_{\mathrm{F}}(\|\mathbf{S}_1^{-1/2}\|_2 + \|\mathbf{S}_2^{-1/2}\|) \\
&+ \|\mathbf{P}_1 - \mathbf{P}_2\|_{\mathrm{F}}\|\mathbf{S}_1^{-1/2}\|_2 + 2\|\mathbf{P}_1 - \mathbf{P}_2\|_{\mathrm{F}}\|\mathbf{S}_1^{-1/2}\|_2 \\
&+ \|\sin\Theta(\mathbf{U}_1, \mathbf{U}_2)\|_2^2\|\mathbf{S}_1^{1/2}\|_{\mathrm{F}} \\
&+ \|\mathbf{W}_{\mathbf{U}}\mathbf{S}_1^{1/2} - \mathbf{S}_2^{1/2}\mathbf{W}_{\mathbf{U}}\|_{\mathrm{F}} \\
\leq{}& \frac{12}{\sigma_0}\sqrt{\frac{d}{n}}\|\mathbf{P}_1 - \mathbf{P}_2\|_{\mathrm{F}} + \sqrt{n}\|\sin\Theta(\mathbf{U}_1, \mathbf{U}_2)\|_2^2 \\
&+ \|\mathbf{W}_{\mathbf{U}}\mathbf{S}_1^{1/2} - \mathbf{S}_2^{1/2}\mathbf{W}_{\mathbf{U}}\|_{\mathrm{F}}.
\end{aligned}$$

By Davis-Kahan theorem, we have, for sufficiently large n,

$$\begin{aligned}
\|\sin\Theta(\mathbf{U}_1, \mathbf{U}_2)\|_2 &\leq \frac{8d\|\mathbf{P}_1 - \mathbf{P}_2\|_{\mathrm{F}}}{\sigma_0^2 n} \leq \frac{8d}{\sigma_0^2 n}\left\{n\rho(\mathbf{X}_1, \mathbf{X}_0) + n\rho(\mathbf{X}_2, \mathbf{X}_0)\right\} \\
&\leq \frac{8M}{\sigma_0^2}\sqrt{\frac{d^3\log n}{n}} \leq \frac{1}{\sqrt{d}},
\end{aligned}$$

as we are assuming that $(d^4\log n)/n \to 0$. Therefore,

$$\sqrt{n}\|\sin\Theta(\mathbf{U}_1, \mathbf{U}_2)\|_2^2 \leq \frac{4d}{\sigma_0^2\sqrt{n}}\|\sin\Theta(\mathbf{U}_1, \mathbf{U}_2)\|_2\|\mathbf{P}_1 - \mathbf{P}_2\|_{\mathrm{F}} \leq \frac{4}{\sigma_0^2}\sqrt{\frac{d}{n}}\|\mathbf{P}_1 - \mathbf{P}_2\|_{\mathrm{F}}.$$

To tackle the last term $\|\mathbf{W}_{\mathbf{U}}\mathbf{S}_1^{1/2} - \mathbf{S}_2^{1/2}\mathbf{W}_{\mathbf{U}}\|_{\mathrm{F}}$, we adopt the technique applied in [30] (see Lemma 2 there) and derive the following decomposition of $\mathbf{W}_{\mathbf{U}}\mathbf{S}_1 - \mathbf{S}_2\mathbf{W}_{\mathbf{U}}$:

$$\begin{aligned}
\mathbf{W}_{\mathbf{U}}\mathbf{S}_1 - \mathbf{S}_2\mathbf{W}_{\mathbf{U}} &= (\mathbf{W}_{\mathbf{U}} - \mathbf{U}_2^{\mathrm{T}}\mathbf{U}_1)\mathbf{S}_1 + \mathbf{U}_2^{\mathrm{T}}\mathbf{U}_1\mathbf{S}_1 - \mathbf{S}_2\mathbf{W}_{\mathbf{U}} \\
&= (\mathbf{W}_{\mathbf{U}} - \mathbf{U}_2^{\mathrm{T}}\mathbf{U}_1)\mathbf{S}_1 + \mathbf{U}_2^{\mathrm{T}}\mathbf{P}_1\mathbf{U}_1 - \mathbf{S}_2\mathbf{W}_{\mathbf{U}} \\
&= (\mathbf{W}_{\mathbf{U}} - \mathbf{U}_2^{\mathrm{T}}\mathbf{U}_1)\mathbf{S}_1 + \mathbf{U}_2^{\mathrm{T}}(\mathbf{P}_1 - \mathbf{P}_2)\mathbf{U}_1 + \mathbf{U}_2^{\mathrm{T}}\mathbf{P}_2\mathbf{U}_1 - \mathbf{S}_2\mathbf{W}_{\mathbf{U}} \\
&= (\mathbf{W}_{\mathbf{U}} - \mathbf{U}_2^{\mathrm{T}}\mathbf{U}_1)\mathbf{S}_1 + \mathbf{U}_2^{\mathrm{T}}(\mathbf{P}_1 - \mathbf{P}_2)\mathbf{U}_1 + \mathbf{S}_2\mathbf{U}_2^{\mathrm{T}}\mathbf{U}_1 - \mathbf{S}_2\mathbf{W}_{\mathbf{U}} \\
&= (\mathbf{W}_{\mathbf{U}} - \mathbf{U}_2^{\mathrm{T}}\mathbf{U}_1)\mathbf{S}_1 + \mathbf{U}_2^{\mathrm{T}}(\mathbf{P}_1 - \mathbf{P}_2)\mathbf{U}_1 + \mathbf{S}_2(\mathbf{U}_2^{\mathrm{T}}\mathbf{U}_1 - \mathbf{W}_{\mathbf{U}}).
\end{aligned}$$

Since by Davis-Kahan theorem,

$$\|\mathbf{W}_{\mathbf{U}} - \mathbf{U}_2^{\mathrm{T}}\mathbf{U}_1\|_2 \leq \|\sin\Theta(\mathbf{U}_1, \mathbf{U}_2)\|_2^2 \leq \frac{8\sqrt{d}}{\sigma_0^2 n}\|\mathbf{P}_1 - \mathbf{P}_2\|_{\mathrm{F}},$$

it follows that

$$\begin{aligned}\|\mathbf{W_U}\mathbf{S}_1 - \mathbf{S}_2\mathbf{W_U}\|_{\mathrm{F}} &\leq \|\mathbf{W_U} - \mathbf{U}_2^{\mathrm{T}}\mathbf{U}_1\|_2(\|\mathbf{S}_1\|_{\mathrm{F}} + \|\mathbf{S}_2\|_{\mathrm{F}}) + \|\mathbf{P}_1 - \mathbf{P}_2\|_{\mathrm{F}} \\ &\leq 2\|\mathbf{P}_1 - \mathbf{P}_2\|_{\mathrm{F}}.\end{aligned}$$

Observe that the (i,j)-th entry of $\mathbf{W_U}\mathbf{S}_1^{1/2} - \mathbf{S}_2^{1/2}\mathbf{W_U}$ can be written as

$$(\mathbf{W_U})_{ij}\left\{\sqrt{\lambda_j(\mathbf{P}_1)} - \sqrt{\lambda_i(\mathbf{P}_2)}\right\} = \frac{(\mathbf{W_U})_{ij}\left\{(\lambda_j(\mathbf{P}_1) - \lambda_i(\mathbf{P}_2)\right\}}{\sqrt{\lambda_j(\mathbf{P}_1)} + \sqrt{\lambda_i(\mathbf{P}_2)}} = \frac{(\mathbf{W_U}\mathbf{S}_1 - \mathbf{S}_2\mathbf{W_U})_{ij}}{\sqrt{\lambda_j(\mathbf{P}_1)} + \sqrt{\lambda_i(\mathbf{P}_2)}},$$

it follows that

$$\|\mathbf{W_U}\mathbf{S}_1^{1/2} - \mathbf{S}_2^{1/2}\mathbf{W_U}\|_{\mathrm{F}} \leq \frac{1}{\sigma_d(\mathbf{X}_1)}\|\mathbf{W_U}\mathbf{S}_1 - \mathbf{S}_2\mathbf{W_U}\|_{\mathrm{F}} \leq \frac{4}{\sigma_0}\sqrt{\frac{d}{n}}\|\mathbf{P}_1 - \mathbf{P}_2\|_{\mathrm{F}}.$$

Combining the above results, we obtain, that for any $\mathbf{X}_1, \mathbf{X}_2 \in \Theta_n$,

$$\|\mathbf{X}_1 - \mathbf{X}_2\mathbf{V}_2\mathbf{W_U}\mathbf{V}_1\|_{\mathrm{F}} \leq \left(\frac{16}{\sigma_0} + \frac{8}{\sigma_0^2}\right)\sqrt{\frac{d}{n}}\|\mathbf{P}_1 - \mathbf{P}_2\|_{\mathrm{F}}. \tag{5.16}$$

Hence,

$$\begin{aligned}\mathcal{F} &= \{\mathbf{X} \in \Theta_n : \rho(\mathbf{X}, \mathbf{X}_0) < \epsilon\} \\ &\subset \left\{\mathbf{X} \in \Theta_n : \inf_{\mathbf{W} \in \mathbb{O}(d)} \|\mathbf{X} - \mathbf{X}_0\mathbf{W}\|_{\mathrm{F}} \leq \left(\frac{16}{\sigma_0} + \frac{8}{\sigma_0^2}\right)\sqrt{nd}\epsilon\right\} \\ &= \bigcup_{\mathbf{W} \in \mathbb{O}(d)} \left\{\mathbf{X} \in \Theta_n : \|\mathbf{X} - \mathbf{X}_0\mathbf{W}\|_{\mathrm{F}} < \left(\frac{16}{\sigma_0} + \frac{8}{\sigma_0^2}\right)\sqrt{nd}\epsilon\right\}.\end{aligned}$$

Now let $\widetilde{\mathcal{O}}(\epsilon)$ be an $\epsilon/\sqrt{d}$-net of $(\mathbb{O}(d), \|\cdot\|_{\mathrm{F}})$. Since $\mathbb{O}(d) \subset \{\mathbf{A} \in \mathbb{R}^{d\times d} : \|\mathbf{A}\|_{\mathrm{F}} = 1\}$, it follows that $|\widetilde{\mathcal{O}}(\epsilon)| \leq (3/\epsilon)^{d^2}$. Therefore, for any

$$\mathbf{X} \in \bigcup_{\mathbf{W} \in \mathbb{O}(d)} \left\{\mathbf{X} \in \Theta_n : \|\mathbf{X} - \mathbf{X}_0\mathbf{W}\|_{\mathrm{F}} < \left(\frac{16}{\sigma_0} + \frac{8}{\sigma_0^2}\right)\sqrt{nd}\epsilon\right\},$$

there exists some $\mathbf{W} \in \mathbb{O}(d)$, and some $\mathbf{R} \in \widetilde{\mathcal{O}}(\epsilon)$, such that $\|\mathbf{W} - \mathbf{R}\|_{\mathrm{F}} \leq \epsilon$, and $\|\mathbf{X} - \mathbf{X}_0\mathbf{W}\|_{\mathrm{F}} < (16/\sigma_0 + 8/\sigma_0^2)\sqrt{nd}\epsilon$, implying that

$$\|\mathbf{X} - \mathbf{X}_0\mathbf{R}\|_{\mathrm{F}} \leq \|\mathbf{X} - \mathbf{X}_0\mathbf{W}\|_{\mathrm{F}} + \|\mathbf{X}_0(\mathbf{W} - \mathbf{R})\|_{\mathrm{F}} < \left(1 + \frac{16}{\sigma_0} + \frac{8}{\sigma_0^2}\right)\sqrt{nd}\epsilon.$$

Hence,

$$\mathbf{X} \in \bigcup_{\mathbf{R}\in\widetilde{\mathcal{O}}(\epsilon)} \left\{\mathbf{X} \in \Theta_n : \|\mathbf{X} - \mathbf{X}_0\mathbf{R}\|_{\mathrm{F}} < \left(1 + \frac{16}{\sigma_0} + \frac{8}{\sigma_0^2}\right)\sqrt{nd}\epsilon\right\},$$

and hence, by the fact that $\rho(\mathbf{X}_1, \mathbf{X}_2) \leq (2/\sqrt{n})\|\mathbf{X}_1 - \mathbf{X}_2\|_{\mathrm{F}}$,

$$\begin{aligned}
&\mathcal{N}\left(\frac{\epsilon}{4}, \mathcal{F}, \rho\right)\\
&\quad\leq \sum_{\mathbf{R}\in\widetilde{\mathcal{O}}(\epsilon)} \mathcal{N}\left[\frac{\epsilon}{4}, \left\{\mathbf{X} \in \Theta_n : \|\mathbf{X} - \mathbf{X}_0\mathbf{R}\|_{\mathrm{F}} < \left(1 + \frac{16}{\sigma_0} + \frac{8}{\sigma_0^2}\right)\sqrt{nd}\epsilon\right\}, \rho\right]\\
&\quad\leq \sum_{\mathbf{R}\in\widetilde{\mathcal{O}}(\epsilon)} \mathcal{N}\left[\frac{\epsilon}{4}, \left\{\mathbf{X} \in \Theta_n : \|\mathbf{X} - \mathbf{X}_0\mathbf{R}\|_{\mathrm{F}} < \left(1 + \frac{16}{\sigma_0} + \frac{8}{\sigma_0^2}\right)\sqrt{nd}\epsilon\right\}, \frac{2}{\sqrt{n}}\|\cdot\|_{\mathrm{F}}\right]\\
&\quad\leq \left(\frac{3}{\epsilon}\right)^{d^2} \mathcal{N}\left[\frac{\sqrt{n}\epsilon}{8}, \left\{\mathbf{X} \in \mathbb{R}^{n\times d} : \|\mathbf{X} - \mathbf{X}_0\mathbf{R}\|_{\mathrm{F}} < \left(1 + \frac{16}{\sigma_0} + \frac{8}{\sigma_0^2}\right)\sqrt{nd}\epsilon\right\}, \|\cdot\|_{\mathrm{F}}\right]\\
&\quad\leq \left(\frac{3}{\epsilon}\right)^{d^2} \left\{24\sqrt{d}\left(1 + \frac{16}{\sigma_0} + \frac{8}{\sigma_0^2}\right)\right\}^{n\times d},
\end{aligned}$$

completing the proof of the first assertion. For the second assertion, we proceed similarly to derive

$$\begin{aligned}
&\{\mathbf{X} \in \Theta_n : \rho(\mathbf{X}, \mathbf{X}_0) \leq 2j\epsilon\}\\
&\quad\subset \bigcup_{\mathbf{W}\in\mathbb{O}(d)} \left\{\mathbf{X} \in \Theta_n : \|\mathbf{X} - \mathbf{X}_0\mathbf{W}\|_{\mathrm{F}} \leq \left(\frac{16}{\sigma_0} + \frac{8}{\sigma_0^2}\right)2\sqrt{nd}j\epsilon\right\}\\
&\quad= \bigcup_{\mathbf{W}\in\widetilde{\mathcal{O}}(\epsilon)} \left\{\mathbf{X} \in \Theta_n : \|\mathbf{X} - \mathbf{X}_0\mathbf{W}\|_{\mathrm{F}} \leq \left(1 + \frac{16}{\sigma_0} + \frac{8}{\sigma_0^2}\right)2\sqrt{nd}j\epsilon\right\}.
\end{aligned}$$

Invoking the fact that $\mathrm{vol}(B_1^d)\sup_{\mathbf{x}\in\mathcal{X}}\pi_{\mathbf{x}} \leq \exp(C_\pi)$ for some constant C_π independent of n, we obtain, for some constant $C > 0$,

$$\begin{aligned}
\Pi\{\mathbf{X} \in \Theta_n : \rho(\mathbf{X}, \mathbf{X}_0) \leq 2j\epsilon\} &\leq \sum_{\mathbf{W}\in\widetilde{\mathcal{O}}(\epsilon)} \Pi\left(\mathbf{X} \in \Theta_n : \|\mathbf{X} - \mathbf{X}_0\mathbf{W}\|_{\mathrm{F}} \leq C\sqrt{nd}j\epsilon\right)\\
&= \sum_{\mathbf{W}\in\widetilde{\mathcal{O}}(\epsilon)} \int\cdots\int_{\{\|\mathbf{X}-\mathbf{X}_0\mathbf{W}\|_{\mathrm{F}}\leq C\sqrt{nd}j\epsilon\}} \prod_{i=1}^n \pi_{\mathbf{x}}(\mathbf{x}_i)\mathrm{d}\mathbf{x}_1\ldots\mathrm{d}\mathbf{x}_n\\
&\leq \left(\frac{3}{\epsilon}\right)^{d^2} \frac{\mathrm{vol}(B_1^{nd})}{\mathrm{vol}(B_1^d)^n}\exp(nC_\pi)\left(C\sqrt{nd}j\epsilon\right)^{nd}\\
&\leq \left(\frac{3}{\epsilon}\right)^{d^2} \exp\left[\{C_\pi - \log\mathrm{vol}(B_1^d)\}n\right]\left(\sqrt{2\pi\mathrm{e}}Cj\epsilon\right)^{nd}.
\end{aligned}$$

The proof is thus completed. □

A coarse posterior contraction result

Theorem 7 in claims that the posterior contraction rate is $1/n$ with respect to $(1/n)\inf_{\mathbf{W}}\|\mathbf{X}-\mathbf{X}_0\mathbf{W}\|_{\mathrm{F}}^2$. It turns out that it is easier to establish a coarser posterior contraction result with an extra logarithmic factor. We show in the following proposition that contraction rate for the edge probability matrix is $\sqrt{(\log n)/n}$ with respect to $(1/n)\|\mathbf{X}\mathbf{X}^{\mathrm{T}}-\mathbf{X}_0\mathbf{X}_0^{\mathrm{T}}\|_{\mathrm{F}}$. Note that Proposition 1 does not imply Theorem 7 but is a weaker result.

To achieve this goal, we need the following local testing lemma tailored for random graph models, which was originally presented [86].

Lemma 17 (Lemma 4.2 in [86]) *Assume that $\mathbf{P}_1, \mathbf{P}_0 \in [0,1]^{n\times n}$ are two distinct edge probability matrices and let $\mathcal{E} = \{\mathbf{P}\in[0,1]^{n\times n} : \|\mathbf{P}-\mathbf{P}_1\|_{\mathrm{F}} \le \|\mathbf{P}_1-\mathbf{P}_0\|_{\mathrm{F}}/2\}$ be a Frobenius ball of radius $\|\mathbf{P}_1-\mathbf{P}_0\|_{\mathrm{F}}/2$ centered at $\mathbf{P}_1$. Based on $A_{ij}\sim\mathrm{Bernoulli}(\mathbf{P}_{ij})$ for $1\le i\le j\le n$, consider testing $H_0 : \mathbf{P} = \mathbf{P}_0$ against $H_A : \mathbf{P}\in\mathcal{E}$. Then there exists a test function ϕ_n, such that*

$$\mathbb{E}_{\mathbf{P}_0}(\phi_n) \le \exp(-C_1\|\mathbf{P}_1-\mathbf{P}_0\|_{\mathrm{F}}^2), \quad \sup_{\mathbf{P}\in\mathcal{E}}\mathbb{E}_{\mathbf{P}}(1-\phi_n) \le \exp(-C_2\|\mathbf{P}_1-\mathbf{P}_0\|_{\mathrm{F}}^2)$$

for some universal constants $C_1, C_2 > 0$ independent of $\mathbf{P}_0, \mathbf{P}_1$, and n.

Proposition 1 *Under the assumption of Theorem 7, there exists some absolute constant $K>0$ and some large constant $M>0$, such that*

$$\mathbb{E}_0\left\{\Pi\left(\frac{1}{n}\|\mathbf{X}\mathbf{X}^{\mathrm{T}}-\mathbf{X}_0\mathbf{X}_0^{\mathrm{T}}\|_{\mathrm{F}} > M\sqrt{\frac{d\log n}{n}}\;\middle|\;\mathbf{A}\right)\right\} \le 3\exp\left(-\frac{1}{2}nd\log n\right)$$

for sufficiently large n.

Proof. Take $\epsilon_n = \sqrt{(d\log n)/n}$. Let $\beta,\gamma>0$ be constants to be determined later. Denote the event Ξ_n to be all $\mathbf{A}$ such that

$$D_n > \exp\left\{(c_\pi + d\log\beta)n - \left(\frac{16\beta^2}{\delta^2}+\gamma\right)nd\log n - nd\left(\log\frac{1}{\epsilon_n}\right)\right\}.$$

Consider the pseudo-metric $\rho : \mathcal{X}^n \times \mathcal{X}^n \to [0, \infty)$ defined by

$$\rho(\mathbf{X}_1, \mathbf{X}_2) = (1/n)\|\mathbf{X}_1\mathbf{X}_1^{\mathrm{T}} - \mathbf{X}_2\mathbf{X}_2^{\mathrm{T}}\|_{\mathrm{F}}.$$

Let $\{\mathbf{X}_1, \ldots, \mathbf{X}_s\}$ be an $\epsilon_n/2$-net of $\{\mathbf{X} \in \mathcal{X}^n : \rho(\mathbf{X}, \mathbf{X}_0) > M\epsilon_n\}$ with metric ρ. Clearly,

$$\rho(\mathbf{X}_1, \mathbf{X}_2) \leq \frac{1}{n}\left\{\|\mathbf{X}_1(\mathbf{X}_1 - \mathbf{X}_2)^{\mathrm{T}}\|_{\mathrm{F}} + \|(\mathbf{X}_1 - \mathbf{X}_2)\mathbf{X}_2^{\mathrm{T}}\|_{\mathrm{F}}\right\} \leq \frac{2}{\sqrt{n}}\|\mathbf{X}_1 - \mathbf{X}_2\|_{\mathrm{F}},$$

implying that

$$\begin{aligned}
s &\leq \mathcal{N}\left(\frac{\epsilon_n}{2}, \mathcal{X}^n, \rho\right) \leq \mathcal{N}\left(\frac{\epsilon_n}{2}, \mathcal{X}^n, \frac{2}{\sqrt{n}}\|\cdot\|_{\mathrm{F}}\right) \\
&\leq \mathcal{N}\left[\frac{\sqrt{n}\epsilon_n}{4}, \{\mathbf{X} \in \mathbb{R}^{n\times d} : \|\mathbf{X}\|_{\mathrm{F}} \leq \sqrt{n}\}, \|\cdot\|_{\mathrm{F}}\right] \leq \left(\frac{12}{\epsilon_n}\right)^{nd}.
\end{aligned}$$

For each $r = 1, \ldots, s$, it can be seen that $\mathbf{X} \in B_\rho(\mathbf{X}_r, \epsilon_n/2)$ implies that $\rho(\mathbf{X}, \mathbf{X}_r) < \epsilon_n/2 \leq M\epsilon_n/2 \leq \rho(\mathbf{X}_r, \mathbf{X}_0)/2$. This allows us to invoke Lemma 17 to construct test functions ϕ_{rn}, $r \in [s]$, such that

$$\begin{aligned}
\mathbb{E}_0\phi_{rn} &\leq \exp\left\{-C_1 n^2\rho^2(\mathbf{X}_r, \mathbf{X}_0)\right\} \leq \exp(-Kn^2M^2\epsilon_n^2) \\
&= \exp(-KM^2 nd\log n), \\
\sup_{\mathbf{X}\in B_\rho(\mathbf{X}_r, \epsilon_n/2)} \mathbb{E}_{\mathbf{X}}(1 - \phi_{rn}) &\leq \exp\left\{-C_2 n^2\rho^2(\mathbf{X}_r, \mathbf{X}_0)\right\} \leq \exp(-Kn^2M^2\epsilon_n^2) \\
&= \exp(-KM^2 nd\log n)
\end{aligned}$$

for some constant $K = \min\{C_1, C_2\}$. Taking $\phi_n = \max_{r\in[s]}\phi_{rn}$ yields the following bounds for the type I and type II error probabilities:

$$\begin{aligned}
\mathbb{E}_0\phi_n &= \mathbb{E}_0\left(\max_{r\in[s]}\phi_{rn}\right) \leq \sum_{r=1}^{s}\mathbb{E}_0(\phi_{rn}) \\
&\leq \exp\left\{nd\log 12 + nd\left(\log\frac{1}{\epsilon_n}\right) - KM^2 nd\log n\right\} \\
&\leq \exp\left\{-\left(KM^2 - 3\right)nd\log n\right\}, \qquad (5.17) \\
\sup_{\mathbf{X}:\rho(\mathbf{X},\mathbf{X}_0)>M\epsilon_n} \mathbb{E}_{\mathbf{X}}(1 - \phi_n) &\leq \max_{r\in[s]}\sup_{\mathbf{X}\in B_\rho(\mathbf{X}_r, \epsilon_n/2)} \mathbb{E}_{\mathbf{X}}\left(1 - \max_{r\in[s]}\phi_{rn}\right) \\
&\leq \max_{r\in[s]}\sup_{\mathbf{X}\in B_\rho(\mathbf{X}_r, \epsilon_n/2)} \mathbb{E}_{\mathbf{X}}(1 - \phi_{rn})
\end{aligned}$$

$$\leq \exp(-KM^2 nd \log n). \tag{5.18}$$

We are now in a position to provide an exponential upper bound for $\mathbb{E}_0 \left[\Pi\{\rho(\mathbf{X}, \mathbf{X}_0) > M\epsilon_n \mid \mathbf{A}\}\right]$:

$$\begin{aligned}
&\mathbb{E}_0 \left[\Pi\{\rho(\mathbf{X}, \mathbf{X}_0) > M\epsilon_n \mid \mathbf{A}\}\right] \\
&\quad \leq \mathbb{E}_0 \left\{ \frac{N_n(\mathbf{X} : \rho(\mathbf{X}, \mathbf{X}_0) > M\epsilon_n)}{D_n} \mathbb{1}(\mathbf{A} \in \Xi_n)(1 - \phi_n) \right\} + \mathbb{E}_0(\phi_n) + \mathbb{P}_0(\Xi_n^c) \\
&\quad \leq \max_{\mathbf{A} \in \Xi_n} \left(\frac{1}{D_n}\right) \mathbb{E}_0 \left\{ (1 - \phi_n) \int_{\{\mathbf{X}:\rho(\mathbf{X},\mathbf{X}_0)>M\epsilon_n\}} \frac{p(\mathbf{A} \mid \mathbf{X})}{p(\mathbf{A} \mid \mathbf{X}_0)} \pi_{\mathbf{X}}(\mathbf{X}) \mathrm{d}\mathbf{X} \right\} \\
&\qquad + \mathbb{E}_0(\phi_n) + \mathbb{P}_0(\Xi_n^c).
\end{aligned}$$

By Fubini's theorem and inequality (5.18), the expected value appearing in the first term of the right-hand side of the above display can be further upper bounded:

$$\begin{aligned}
&\mathbb{E}_0 \left\{ (1 - \phi_n) \int_{\{\mathbf{X}:\rho(\mathbf{X},\mathbf{X}_0)>M\epsilon_n\}} \frac{p(\mathbf{A} \mid \mathbf{X})}{p(\mathbf{A} \mid \mathbf{X}_0)} \pi_{\mathbf{X}}(\mathbf{X}) \mathrm{d}\mathbf{X} \right\} \\
&\quad = \int_{\{\mathbf{X}:\rho(\mathbf{X},\mathbf{X}_0)>M\epsilon_n\}} \mathbb{E}_0 \left\{ (1 - \phi_n) \frac{p(\mathbf{A} \mid \mathbf{X})}{p(\mathbf{A} \mid \mathbf{X}_0)} \right\} \pi_{\mathbf{X}}(\mathbf{X}) \mathrm{d}\mathbf{X} \\
&\quad \leq \int \sup_{\{\mathbf{X}:\rho(\mathbf{X},\mathbf{X}_0)>M\epsilon_n\}} \mathbb{E}_{\mathbf{X}}\{(1 - \phi_n)\} \pi_{\mathbf{X}}(\mathbf{X}) \mathrm{d}\mathbf{X} \leq \exp(-KM^2 nd \log n).
\end{aligned}$$

Hence, invoking Lemma 15 and inequality (5.17) and setting $\beta = \delta^2$, $\gamma = 8$, we have, for some constant $c(\delta)$ depending only on δ, that

$$\begin{aligned}
&\mathbb{E}_0 \left[\Pi\{\rho(\mathbf{X}, \mathbf{X}_0) > M\epsilon_n \mid \mathbf{A}\}\right] \\
&\quad \leq \exp\left\{ -(c_\pi + d \log \delta^2) n + (16\delta^2 + 10) nd \log n - KM^2 nd \log n \right\} \\
&\qquad + \exp\left\{ -\left(KM^2 - 3\right) nd \log n \right\} + \exp\left(-\frac{nd \log n}{2} \right) \\
&\quad \leq 2 \exp\left[-\left\{ KM^2 - c(\delta) \right\} nd \log n \right] + \exp\left(-\frac{nd \log n}{2} \right).
\end{aligned}$$

Taking M sufficiently large such that $KM^2 - c(\delta) > 1/2$ completes the proof. $\square$

Refinement of posterior contraction by restriction

We now refine the contraction rate in Proposition 1 but the restriction of the posterior distribution over the set $\{(1/n)\|\mathbf{X}\mathbf{X}^{\mathrm{T}} - \mathbf{X}_0\mathbf{X}_0^{\mathrm{T}}\|_{\mathrm{F}} \leq M\sqrt{(\log n)/n}\}$. This will require the use of the following global testing Lemma, which was originally presented in [87], and is adapted to the random dot product graph model for our purpose.

Lemma 18 (Lemma 9 in [87]) *Let* $\mathbf{A} \sim \mathrm{RDPG}(\mathbf{X})$ *for some* $\mathbf{X} \in \mathcal{X}^n$*. Define a pseudo-metric* $\rho : \mathbb{R}^{n\times d} \times \mathbb{R}^{n\times d} \to [0, \infty)$ *by* $\rho(\mathbf{X}, \mathbf{X}_0) = (1/n)\|\mathbf{X}\mathbf{X}^{\mathrm{T}} - \mathbf{X}_0\mathbf{X}_0^{\mathrm{T}}\|_{\mathrm{F}}$*. Let* $\Theta_n \subset \mathcal{X}^n$ *be a collection of* $n \times d$ *matrices that lie in* $\mathcal{X}^n$*. Suppose that for some non-increasing function* $\epsilon \mapsto N(\epsilon)$*, and some* $\epsilon_n \geq 0$*,*

$$\mathcal{N}\left[\frac{\epsilon}{4}, \{\mathbf{X} \in \Theta_n : \rho(\mathbf{X}, \mathbf{X}_0) < \epsilon\}, \rho\right] \leq N(\epsilon) \quad \text{for all } \epsilon > \epsilon_n.$$

Then for $K = \min\{C_1, C_2\}$ *appearing in Lemma 17, and for any* $\epsilon > \epsilon_n$*, there exists a test function* ϕ_n *for testing testing* $H_0 : \mathbf{X} = \mathbf{X}_0$ *versus* $H_A : \mathbf{X} \in \Theta_n, \rho(\mathbf{X}, \mathbf{X}_0) > j\epsilon$ *that depends on* ϵ*, such that for every* $j \in \mathbb{N}$*,*

$$\mathbb{E}_0(\phi_n) \leq N(\epsilon)\frac{\exp(-Kn^2\epsilon^2)}{1 - \exp(-Kn^2\epsilon^2)}, \qquad \sup_{\mathbf{X}\in\Theta_n, \rho(\mathbf{X},\mathbf{X}_0) > j\epsilon} \mathbb{E}_{\mathbf{X}}(1 - \phi_n) \leq \exp(-Kn^2\epsilon^2 j^2).$$

Proof of Theorem 7. Denote the target posterior contraction rate $\epsilon_n = n^{-1/2}$. Consider the pseudo-metric $\rho(\mathbf{X}_1, \mathbf{X}_2) = (1/n)\|\mathbf{X}_1\mathbf{X}_1^{\mathrm{T}} - \mathbf{X}_2\mathbf{X}_2^{\mathrm{T}}\|_{\mathrm{F}}$. By Proposition 1, we can take $\Theta_n = \{\mathbf{X} \in \mathcal{X}^n : \rho(\mathbf{X}, \mathbf{X}_0) < M\sqrt{(d\log n)/n}\}$ for some large constant $M > 0$ such that

$$\mathbb{E}_0\{\Pi\left(\Theta_n \mid \mathbf{A}\right)\} \leq 3\exp\left(-\frac{1}{2}nd\log n\right).$$

The proof is based on a refinement of that of Theorem 1 in [87] with exponential error bound. We breakdown the proof into the following components.

- **Component 1.** By the first assertion of Lemma 16, we have, for some constant $m > 0$ to be determined later,

$$\begin{aligned}&\sup_{\epsilon > m\epsilon_n} \log\mathcal{N}\left[\frac{\epsilon}{4}, \{\mathbf{X} \in \Theta_n : \rho(\mathbf{X}, \mathbf{X}_0) < \epsilon\}, \rho\right] \\ &\quad\leq d^2\left(\log\frac{3}{m\epsilon_n}\right) + nd\log\left\{24\sqrt{d}\left(1 + \frac{16}{\sigma_0} + \frac{8}{\sigma_0^2}\right)\right\} \leq Ln^2\epsilon_n^2\end{aligned}$$

for some constants $L > 0$. Invoking Lemma 18, we obtain some test function ϕ_n, such that for any $j \in \mathbb{N}_+$,

$$\mathbb{E}_0(\phi_n) \le \frac{\exp\{-(Km^2 - L)n\}}{1 - \exp(-Km^2 n)}, \quad \sup_{\mathbf{X}\in\Theta_n:\rho(\mathbf{X},\mathbf{X}_0)>jm\epsilon_n} \mathbb{E}_{\mathbf{X}}(1-\phi_n) \le \exp(-Kj^2m^2n).$$

The first type I error probability bound appearing in the last display immediately implies

$$\mathbb{E}_0\left[\Pi\left\{\mathbf{X}\in\Theta_n : \rho(\mathbf{X},\mathbf{X}_0) \ge m\epsilon_n\right\}\phi_n\right] \le \frac{\exp\{-(Km^2 - L)n\}}{1 - \exp(-Km^2 n)} \tag{5.19}$$

for any $J \ge 1$.

- **Component 2.** Invoking the aforementioned type II error probability bound in the derivation of the first component, Fubini's theorem, and the second assertion of Lemma 16 leads to

$$\begin{aligned}
&\mathbb{E}_0\left[N_n\{\mathbf{X}\in\Theta_n : mj\epsilon_n < \rho(\mathbf{X},\mathbf{X}_0) \le m(j+1)\epsilon_n\}(1-\phi_n)\right] \\
&= \int_{\{\mathbf{X}\in\Theta_n : mj\epsilon_n < \rho(\mathbf{X},\mathbf{X}_0)\le m(j+1)\epsilon_n\}} \mathbb{E}_0\left\{(1-\phi_n)\frac{p(\mathbf{A}\mid\mathbf{X})}{p(\mathbf{A}\mid\mathbf{X}_0)}\right\}\Pi(\mathrm{d}\mathbf{X}) \\
&\le \Pi\left\{\mathbf{X}\in\Theta_n : \rho(\mathbf{X},\mathbf{X}_0) \le m(j+1)\epsilon_n\right\} \sup_{\mathbf{X}\in\Theta_n:\rho(\mathbf{X},\mathbf{X}_0)>mj\epsilon_n} \mathbb{E}_{\mathbf{X}}(1-\phi_n) \\
&\le \exp\left\{-\left(\frac{K}{2}j^2m^2 - d\log m - C\right)n\right\}(\epsilon_n)^{nd}.
\end{aligned}$$

for some constant $C > 0$. Letting Ξ_n to be the event of all $\mathbf{A}$ such that

$$D_n > \exp\left\{-\left(16\delta^2 + \gamma - c_\pi - d\log\delta^2\right)n\right\}(\epsilon_n)^{nd},$$

we further obtain for some constant $\widetilde{C} > 0$ that

$$\begin{aligned}
&\mathbb{E}_0\left[\Pi\left\{\mathbf{X}\in\Theta_n : mj\epsilon_n < \rho(\mathbf{X},\mathbf{X}_0) \le mj\epsilon_n + m\epsilon_n\right\}(1-\phi_n)\mathbb{1}(\mathbf{A}\in\Xi_n)\right] \\
&\quad \le \max_{\mathbf{A}\in\Xi_n}\left(\frac{1}{D_n}\right)\mathbb{E}_0\left[N_n\{\mathbf{X}\in\Theta_n : mj\epsilon_n < \rho(\mathbf{X},\mathbf{X}_0) \le mj\epsilon_n + m\epsilon_n\}(1-\phi_n)\right] \\
&\quad \le \exp\left\{-\left(\frac{K}{2}j^2m^2 - d\log m - \widetilde{C}\right)n\right\},
\end{aligned} \tag{5.20}$$

and Lemma 15 allows us to control the probability of Ξ_n^c with $\gamma = 8$ and $\beta = \delta^2$:

$$\mathbb{P}_0\left(\Xi_n^c\right) \le \exp\left(-\frac{nd}{2}\right). \tag{5.21}$$

We now decompose $\mathbb{E}_0[\Pi\{\mathbf{X} \in \Theta_n : \rho(\mathbf{X}, \mathbf{X}_0) > m\epsilon_n \mid \mathbf{A}\}]$ as follows:

$$\begin{aligned}
&\mathbb{E}_0[\Pi\{\mathbf{X} \in \Theta_n : \rho(\mathbf{X}, \mathbf{X}_0) > m\epsilon_n \mid \mathbf{A}\}] \\
&\quad \leq \mathbb{E}_0[\Pi\{\mathbf{X} \in \Theta_n : \rho(\mathbf{X}, \mathbf{X}_0) > m\epsilon_n \mid \mathbf{A}\}(1-\phi_n)\mathbb{1}(\mathbf{A} \in \Xi_n)] \\
&\quad\quad + \mathbb{P}_0(\Xi_n^c) + \mathbb{E}_0[\Pi\{\mathbf{X} \in \Theta_n : \rho(\mathbf{X}, \mathbf{X}_0) > m\epsilon_n \mid \mathbf{A}\}\phi_n].
\end{aligned}$$

Observe that by taking a sufficiently large m such that $Km^2/2 - d\log m - \widetilde{C} > KJm^2/4$, we have, by inequality (5.20),

$$\begin{aligned}
&\mathbb{E}_0[\Pi\{\mathbf{X} \in \Theta_n : \rho(\mathbf{X}, \mathbf{X}_0) > m\epsilon_n \mid \mathbf{A}\}(1-\phi_n)\mathbb{1}(\mathbf{A} \in \Xi_n)] \\
&\quad \leq \sum_{j=1}^{\infty} \mathbb{E}_0[\Pi\{\mathbf{X} \in \Theta_n : mj\epsilon_n < \rho(\mathbf{X}, \mathbf{X}_0) \leq m(j+1)\epsilon_n \mid \mathbf{A}\}(1-\phi_n)\mathbb{1}(\mathbf{A} \in \Xi_n)] \\
&\quad \leq \exp\left\{(d\log m + \widetilde{C})\right\} \sum_{j=1}^{\infty} \exp\left(-\frac{K}{2}j^2m^2n\right) \\
&\quad \leq \exp\left\{(d\log m + \widetilde{C})n\right\} \sum_{j=1}^{\infty} \exp\left(-\frac{K}{2}jm^2n\right) \\
&\quad \leq \frac{\exp\left\{-(Km^2/2 - d\log m - \widetilde{C})n\right\}}{1 - \exp(-Km^2n/2)} \\
&\quad \leq \frac{\exp(-Km^2n/4)}{1 - \exp(-Km^2n/2)}.
\end{aligned}$$

It follows from inequalities (5.19) and (5.21) that

$$\mathbb{E}_0[\Pi\{\mathbf{X} \in \Theta_n : \rho(\mathbf{X}, \mathbf{X}_0) > m\epsilon_n \mid \mathbf{A}\}] \leq \frac{2\exp(-Km^2n/4)}{1 - \exp(-Km^2n/2)} + \exp\left(-\frac{nd}{2}\right)$$

by further requiring $Km^2 - L \geq Km^2/4$. Hence, we invoke Proposition 1 to draw the following conclusion: there exists some large constants M_1, M and an absolute constant $K > 0$, such that for sufficiently large n,

$$\begin{aligned}
&\mathbb{E}_0\left[\Pi\left\{\rho(\mathbf{X}, \mathbf{X}_0) > M_1\epsilon_n \mid \mathbf{A}\right\}\right] \\
&\quad \leq \mathbb{E}_0\left[\Pi\left\{\mathbf{X} \in \Theta_n : \rho(\mathbf{X}, \mathbf{X}_0) > M_1\epsilon_n \mid \mathbf{A}\right\}\right] + \mathbb{E}_0\left\{\Pi(\Theta_n \mid \mathbf{A})\right\} \\
&\quad \leq 4\exp\left(-\frac{KM_1^2n}{4}\right) + \exp\left(-\frac{nd}{2}\right) + 3\exp\left(-\frac{1}{2}nd\log n\right) \leq 8\exp\left(-\frac{1}{2}nd\right).
\end{aligned}$$

Namely, there exists some constant $C_0 > 0$ that is independent of n, such that

$$\mathbb{E}_0\left[\Pi\left\{\rho(\mathbf{X}, \mathbf{X}_0) > M_1\epsilon_n \mid \mathbf{A}\right\}\right] \leq 8\exp\left(-\frac{1}{2}nd\right)$$

for sufficiently large $M_1 > 0$. The proof of the first assertion is thus completed. The second assertion directly follows from the following observation: We see from the proof of Proposition 1 (see inequality (5.16)) that for any $\mathbf{X} \in \Theta_n$, there exists an orthogonal $\mathbf{W}^* \in \mathbb{O}(d)$, such that

$$\frac{1}{\sqrt{n}} \inf_{\mathbf{W}\in\mathbb{O}(d)} \|\mathbf{X} - \mathbf{X}_0\mathbf{W}(\mathbf{X}, \mathbf{X}_0)\|_{\mathrm{F}} \leq \frac{1}{\sqrt{n}}\|\mathbf{X} - \mathbf{X}_0\mathbf{W}^*\|_{\mathrm{F}} \lesssim \rho(\mathbf{X}, \mathbf{X}_0).$$

$\square$

Proof of Theorem 8. Before proving the two assertions of the theorem, we first show that $\widetilde{\mathbf{P}}$ is close to $\mathbf{P}_0 = \mathbf{X}_0\mathbf{X}_0^{\mathrm{T}}$ in mean-squared error. Take the pseudo-metric $\rho(\mathbf{X}_1, \mathbf{X}_2) = (1/n)\|\mathbf{X}_1\mathbf{X}_1^{\mathrm{T}} - \mathbf{X}_2\mathbf{X}_2^{\mathrm{T}}\|_{\mathrm{F}}$. Let M_1 and M_2 be the constants provided by Theorem 7. By the Jensen's inequality, we have, by Theorem 7, that

$$\begin{aligned}
\mathbb{E}_0\left(\frac{1}{n^2}\|\widetilde{\mathbf{P}} - \mathbf{X}_0\mathbf{X}_0^{\mathrm{T}}\|_{\mathrm{F}}^2\right) &\leq \mathbb{E}_0\left\{\frac{1}{n^2}\int_{\mathcal{X}^n}\|\mathbf{X}\mathbf{X}^{\mathrm{T}} - \mathbf{X}_0\mathbf{X}_0\|_{\mathrm{F}}^2\Pi(\mathrm{d}\mathbf{X} \mid \mathbf{A})\right\}\\
&\leq \mathbb{E}_0\left\{\frac{1}{n^2}\int_{\{\rho(\mathbf{X},\mathbf{X}_0)\leq M_1/\sqrt{n}\}}\|\mathbf{X}\mathbf{X}^{\mathrm{T}} - \mathbf{X}_0\mathbf{X}_0\|_{\mathrm{F}}^2\Pi(\mathrm{d}\mathbf{X} \mid \mathbf{A})\right\}\\
&\quad + \mathbb{E}_0\left[\Pi\left\{\rho(\mathbf{X}, \mathbf{X}_0) > \frac{M_1}{\sqrt{n}}\,\middle|\, \mathbf{A}\right\}\right]\left(\frac{4}{n^2}\max_{\mathbf{X}\in\mathcal{X}^n}\|\mathbf{X}\|_{\mathrm{F}}^4\right)\\
&\leq \frac{M_1^2}{n} + 32\exp\left(-\frac{nd}{2}\right) \leq \frac{2M_1^2}{n}
\end{aligned}$$

For the first assertion, we adopt the technique applied in the proof of Theorem 7. Let $\mathbf{X}_0$ yield singular value decompositions $\mathbf{X}_0 = \mathbf{U}_0\mathbf{S}_0^{1/2}\mathbf{V}_0^{\mathrm{T}}$, where $\mathbf{U}_0 \in \mathbb{O}(n, d)$ and $\mathbf{V}_0 \in \mathbb{O}(d)$. Further let $\mathbf{U}_0^{\mathrm{T}}\widehat{\mathbf{U}} = \mathbf{W}_1\mathbf{\Sigma}\mathbf{W}_2^{\mathrm{T}}$ be the singular value decomposition of $\mathbf{U}_0^{\mathrm{T}}\widehat{\mathbf{U}}$, and let $\mathbf{W}_{\mathbf{U}} = \mathbf{W}_1\mathbf{W}_2^{\mathrm{T}}$. Denote $\mathbf{P}_0 = \mathbf{X}_0\mathbf{X}_0^{\mathrm{T}}$. Then by Lemma 13 and the fact that $\|\mathbf{A}\mathbf{B}\|_{\mathrm{F}} \leq \|\mathbf{A}\|_{\mathrm{F}}\|\mathbf{B}\|_2$, over the event

$$\Gamma_n = \bigcap_{k=1}^{d}\left\{|\sigma_k^2(\widehat{\mathbf{X}}) - \sigma_k^2(\mathbf{X}_0)| \leq \frac{n\lambda_k(\mathbf{\Delta})}{4}\right\},$$

we have,

$$\|\widehat{\mathbf{X}} - \mathbf{X}_0\mathbf{V}_0\mathbf{W}_{\mathbf{U}}\|_{\mathrm{F}} \leq \|\mathbf{S}_0^{-1/2}\|_2\|\widetilde{\mathbf{P}} - \mathbf{P}_0\|_{\mathrm{F}} + \|\widetilde{\mathbf{P}} - \mathbf{P}_0\|_{\mathrm{F}}(\|\widehat{\mathbf{S}}^{-1/2}\|_2 + \|\mathbf{S}_0^{-1/2}\|)$$

$$
\begin{aligned}
&+ \|\widetilde{\mathbf{P}} - \mathbf{P}_0\|_{\mathrm{F}}\|\widehat{\mathbf{S}}^{-1/2}\|_2 + 2\|\widetilde{\mathbf{P}} - \mathbf{P}_0\|_{\mathrm{F}}\|\widehat{\mathbf{S}}^{-1/2}\|_2\\
&+ \|\sin\Theta(\widehat{\mathbf{U}}, \mathbf{U}_0)\|_2^2\|\widehat{\mathbf{S}}^{1/2}\|_{\mathrm{F}} + \|\mathbf{W}_{\mathbf{U}}\widehat{\mathbf{S}}^{1/2} - \mathbf{S}_0^{1/2}\mathbf{W}_{\mathbf{U}}\|_{\mathrm{F}}\\
\leq{}& \frac{12}{\sqrt{n\lambda_d(\mathbf{\Delta})}}\|\widetilde{\mathbf{P}} - \mathbf{P}_0\|_{\mathrm{F}} + \sqrt{\frac{n\lambda_d(\mathbf{\Delta})}{4}}\|\sin\Theta(\widehat{\mathbf{U}}, \mathbf{U}_0)\|_2\\
&+ \|\mathbf{W}_{\mathbf{U}}\widehat{\mathbf{S}}^{1/2} - \mathbf{S}_0^{1/2}\mathbf{W}_{\mathbf{U}}\|_{\mathrm{F}}.
\end{aligned}
$$

By Davis-Kahan theorem, $\|\sin\Theta(\widehat{\mathbf{U}}, \mathbf{U}_0)\|_2 \lesssim \|\widetilde{\mathbf{P}} - \mathbf{P}_0\|_{\mathrm{F}}/n\lambda_d(\mathbf{\Delta})$. The last term $\|\mathbf{W}_{\mathbf{U}}\widehat{\mathbf{S}}^{1/2} - \mathbf{S}_0^{1/2}\mathbf{W}_{\mathbf{U}}\|_{\mathrm{F}}$ can be upper bounded by $2\|\mathbf{W}_{\mathbf{U}}\widehat{\mathbf{S}} - \mathbf{S}_0\mathbf{W}_{\mathbf{U}}\|_{\mathrm{F}}/\sqrt{n\lambda_d(\mathbf{\Delta})}$, and $\mathbf{W}_{\mathbf{U}}\widehat{\mathbf{S}} - \mathbf{S}_0\mathbf{W}_{\mathbf{U}}$ can be decomposed as follows:

$$
\mathbf{W}_{\mathbf{U}}\widehat{\mathbf{S}} - \mathbf{S}_0\mathbf{W}_{\mathbf{U}} = (\mathbf{W}_{\mathbf{U}} - \mathbf{U}_0^{\mathrm{T}}\widehat{\mathbf{U}})\widehat{\mathbf{S}} + \mathbf{U}_0^{\mathrm{T}}(\widetilde{\mathbf{P}} - \mathbf{P}_0)\widehat{\mathbf{U}} + \mathbf{S}_0(\mathbf{U}_0^{\mathrm{T}}\widehat{\mathbf{U}} - \mathbf{W}_{\mathbf{U}}).
$$

Since by Davis-Kahan theorem,

$$
\|\mathbf{W}_{\mathbf{U}} - \mathbf{U}_0^{\mathrm{T}}\widehat{\mathbf{U}}\|_2 \leq \|\sin\Theta(\widehat{\mathbf{U}}, \mathbf{U}_0)\|_2^2 \lesssim \frac{\|\widetilde{\mathbf{P}} - \mathbf{P}_0\|_{\mathrm{F}}}{n},
$$

it follows that $\|\mathbf{W}_{\mathbf{U}}\widehat{\mathbf{S}} - \mathbf{S}_0\mathbf{W}_{\mathbf{U}}\|_{\mathrm{F}} \leq \|\mathbf{W}_{\mathbf{U}} - \mathbf{U}_0^{\mathrm{T}}\widehat{\mathbf{U}}\|_2(\|\widehat{\mathbf{S}}\|_{\mathrm{F}} + \|\mathbf{S}_0\|_{\mathrm{F}}) + \|\widetilde{\mathbf{P}} - \mathbf{P}_0\|_{\mathrm{F}} \lesssim \|\widetilde{\mathbf{P}} - \mathbf{P}_0\|_{\mathrm{F}}$, and that

$$
\|\mathbf{W}_{\mathbf{U}}^{1/2}\widehat{\mathbf{S}}^{1/2} - \mathbf{S}_0^{1/2}\mathbf{W}_{\mathbf{U}}\|_{\mathrm{F}} \lesssim \frac{1}{\sqrt{n}}\|\widehat{\mathbf{P}} - \mathbf{P}_0\|_{\mathrm{F}}.
$$

Combining the above results, over the event Γ_n, we obtain $\|\widehat{\mathbf{X}} - \mathbf{X}_0\mathbf{V}_0\mathbf{W}_{\mathbf{U}}\|_{\mathrm{F}} \leq (C_{\mathbf{\Delta}}/\sqrt{n})\|\widetilde{\mathbf{P}} - \mathbf{P}_0\|_{\mathrm{F}}$ for some constant $C_{\mathbf{\Delta}}$ depend on $\mathbf{\Delta}$. Also note that the probability of Γ_n^c can also be bounded by Markov's inequality,

$$
\begin{aligned}
\mathbb{P}_0(\Gamma_n^c) &= \mathbb{P}_0\left[\bigcup_{k=1}^{d}\left\{|\sigma_k^2(\widehat{\mathbf{X}}) - \sigma_k^2(\mathbf{X}_0)| \geq \frac{n\lambda_k(\mathbf{\Delta})}{4}\right\}\right]\\
&\leq \mathbb{P}_0\left\{\frac{1}{n}\|\widetilde{\mathbf{P}} - \mathbf{P}_0\|_{\mathrm{F}} > \frac{\lambda_d(\mathbf{\Delta})}{4}\right\} \lesssim \frac{1}{n}.
\end{aligned}
$$

Observe that

$$
\|\widehat{\mathbf{X}}\|_{\mathrm{F}}^2 \leq \|\widetilde{\mathbf{P}}\|_{\mathrm{F}} \leq \int_{\mathcal{X}^n}\|\mathbf{X}\mathbf{X}^{\mathrm{T}}\|_{\mathrm{F}}\Pi(\mathrm{d}\mathbf{X} \mid \mathbf{A}) \leq \int_{\mathcal{X}^n}\|\mathbf{X}\|_{\mathrm{F}}^2\Pi(\mathrm{d}\mathbf{X} \mid \mathbf{A}) \leq n.
$$

Therefore,

$$
\mathbb{E}_0\left(\frac{1}{n}\inf_{\mathbf{W}\in\mathbb{O}(d)}\|\widehat{\mathbf{X}} - \mathbf{X}_0\mathbf{W}\|_{\mathrm{F}}^2\right) \leq \mathbb{E}_0\left\{\mathbb{1}(\mathbf{A} \in \Gamma_n)\frac{1}{n}\|\widehat{\mathbf{X}} - \mathbf{X}_0\mathbf{V}_0\mathbf{W}_{\mathbf{U}}\|_{\mathrm{F}}^2\right\} + 2\mathbb{P}_0(\Gamma_n^c)
$$

$$\leq \mathbb{E}_0\left(\frac{C_{\mathbf{\Delta}}^2}{n}\|\widetilde{\mathbf{P}}-\mathbf{P}_0\|_{\mathrm{F}}^2\right)+2\mathbb{P}_0\left(\Gamma_n^c\right)\lesssim\frac{1}{n}.$$

Now we focus on the second assertion. For convenience denote $\mathbf{E}=\widetilde{\mathbf{P}}-\mathbf{X}_0\mathbf{X}_0^{\mathrm{T}}$. Then for any $t>0$, the moment generating function of $\|\mathbf{E}\|_{\mathrm{F}}$ satisfies

$$\begin{aligned}
M_{\|\mathbf{E}\|_{\mathrm{F}}}(t) &= \mathbb{E}_0\left[\exp\left\{t\left\|\int_{\mathcal{X}^n}(\mathbf{X}\mathbf{X}^{\mathrm{T}}-\mathbf{X}_0\mathbf{X}_0^{\mathrm{T}})\Pi(\mathrm{d}\mathbf{X}\mid\mathbf{A})\right\|_{\mathrm{F}}\right\}\right]\\
&\leq \mathbb{E}_0\left[\exp\left\{t\int_{\mathcal{X}^n}\left\|\mathbf{X}\mathbf{X}^{\mathrm{T}}-\mathbf{X}_0\mathbf{X}_0^{\mathrm{T}}\right\|_{\mathrm{F}}\Pi(\mathrm{d}\mathbf{X}\mid\mathbf{A})\right\}\right]\\
&\leq \mathbb{E}_0\left[\exp\left\{t\int_{\{\|\mathbf{X}\mathbf{X}^{\mathrm{T}}-\mathbf{X}_0\mathbf{X}_0^{\mathrm{T}}\|_{\mathrm{F}}\leq\sqrt{n}M_1\}}\left\|\mathbf{X}\mathbf{X}^{\mathrm{T}}-\mathbf{X}_0\mathbf{X}_0^{\mathrm{T}}\right\|_{\mathrm{F}}\Pi(\mathrm{d}\mathbf{X}\mid\mathbf{A})\right\}\right]\\
&\quad+\mathbb{E}_0\left\{\Pi\left(\|\mathbf{X}\mathbf{X}^{\mathrm{T}}-\mathbf{X}_0\mathbf{X}_0^{\mathrm{T}}\|_{\mathrm{F}}>\sqrt{n}M_1\mid\mathbf{A}\right)\right\}\\
&\qquad\times\exp\left\{t\sup_{\mathbf{X}\in\mathcal{X}^n}\|\mathbf{X}\mathbf{X}^{\mathrm{T}}-\mathbf{X}_0\mathbf{X}_0^{\mathrm{T}}\|_{\mathrm{F}}\right\}\\
&\leq \exp\left(tM_1\sqrt{n}\right)+8\exp\left(2nt-\frac{nd}{2}\right).
\end{aligned}$$

It follows from Chernoff bound with $t=d/4$ that,

$$\mathbb{P}_0\left(\|\mathbf{E}\|_{\mathrm{F}}>2M_1\sqrt{n}\right)\leq 2\exp\left(-\frac{1}{4}M_1d\sqrt{n}\right)$$

for sufficiently large n. The proof is completed by observing that by Davis-Kahan theorem,

$$\|\widehat{\mathbf{U}}-\mathbf{U}_0\mathbf{W}_{\mathbf{U}}\|_{\mathrm{F}}\leq\sqrt{2d}\|\sin\Theta(\widehat{\mathbf{U}},\mathbf{U}_0)\|_2\leq\frac{2\sqrt{2d}\|\mathbf{E}\|_{\mathrm{F}}}{\lambda_d(\mathbf{X}_0\mathbf{X}_0^{\mathrm{T}})}\leq\frac{4\sqrt{2d}\|\mathbf{E}\|_{\mathrm{F}}}{n\lambda_d(\mathbf{\Delta})}.$$

□

5.2.4 Proofs for Section 3.4

Proof of Theorem 10. Assume that *a posteriori* the event $\{\|\mathbf{X}-\mathbf{X}_0\mathbf{W}(\mathbf{X},\mathbf{X}_0)\|_{\mathrm{F}}\leq M_2\}$ occurs, where $\mathbf{W}(\mathbf{X},\mathbf{X}_0)=\arg\inf_{\mathbf{W}\in\mathbb{O}(d)}\|\mathbf{X}-\mathbf{X}_0\mathbf{W}\|_{\mathrm{F}}$. Observe that by definition and triangle inequality,

$$\begin{aligned}
\|\mathbf{C}(\mathbf{X})-\mathbf{X}_0\mathbf{W}(\mathbf{X}_0,\mathbf{X})\|_{\mathrm{F}} &\leq \|\mathbf{C}(\mathbf{X})-\mathbf{X}\|_{\mathrm{F}}+\|\mathbf{X}-\mathbf{X}_0\mathbf{W}(\mathbf{X}_0,\mathbf{X})\|_{\mathrm{F}}\\
&\leq 2\|\mathbf{X}-\mathbf{X}_0\mathbf{W}(\mathbf{X}_0,\mathbf{X})\|_{\mathrm{F}}\leq 2M_2.
\end{aligned}$$

Now we argue that the number of rows $\mathcal{V} = \{i \in [n] : \|\{\mathbf{C}(\mathbf{X})\}_{i*} - \mathbf{W}(\mathbf{X}_0, \mathbf{X})^{\mathrm{T}}\mathbf{x}_{0i}\|_2 > \xi/2\}$ is no greater than $16M_2^2/\xi^2$ by contradiction. Assuming otherwise, then we obtain

$$\|\mathbf{C}(\mathbf{X}) - \mathbf{X}_0\mathbf{W}(\mathbf{X}_0, \mathbf{X})\|_{\mathrm{F}}^2 > \left(\frac{16M_2^2}{\xi^2}\right)\left(\frac{\xi}{2}\right)^2 = 4M_2^2,$$

contradicting with the previous observation. Namely, $|\mathcal{V}^c| \geq n - 16M^2/\xi^2$. Consequently, for any $i, j \in \mathcal{V}^c$, $\{\mathbf{C}(\mathbf{X})\}_{i*} = \{\mathbf{C}(\mathbf{X})\}_{j*}$, we see that

$$\begin{aligned}\|\mathbf{x}_{0i} - \mathbf{x}_{0j}\|_2 &= \|\mathbf{W}(\mathbf{X}_0, \mathbf{X})^{\mathrm{T}}(\mathbf{x}_{0i} - \mathbf{x}_{0j})\|_2 \\ &\leq \|\{\mathbf{C}(\mathbf{X})\}_{i*} - \mathbf{W}(\mathbf{X}_0, \mathbf{X})^{\mathrm{T}}\mathbf{x}_{0i}\|_2 + \|\{\mathbf{C}(\mathbf{X})\}_{j*} - \mathbf{W}(\mathbf{X}_0, \mathbf{X})^{\mathrm{T}}\mathbf{x}_{0j}\|_2 \leq \xi,\end{aligned}$$

implying that $\mathbf{x}_{0i} = \mathbf{x}_{0j}$ by assumption. Note that $n_k \geq |\mathcal{V}|$ for all k, *i.e.*, $\{\mathbf{x}_{0i} : i \in \mathcal{V}^c\} = \{\mathbf{x}_{0k}^* : k \in [K]\}$, it follows that for each $k \in [K]$, $B_{\|\cdot\|_2}\{\mathbf{W}(\mathbf{X}_0, \mathbf{X})^{\mathrm{T}}\mathbf{x}_{0k}^*, \xi/2\}$ contains at least one element of $[\{\mathbf{C}(\mathbf{X})\}_{i*} : i \in \mathcal{V}^c]$. Since $B_{\|\cdot\|_2}\{\mathbf{W}(\mathbf{X}_0, \mathbf{X})^{\mathrm{T}}\mathbf{x}_{0k}^*, \xi/2\}$ are disjoint by assumption, and there are only K distinct rows in $\mathbf{C}(\mathbf{X})$, it follows directly from the pigeonhole principle that each $B_{\|\cdot\|_2}\{\mathbf{W}(\mathbf{X}_0, \mathbf{X})^{\mathrm{T}}\mathbf{x}_{0k}^*, \xi/2\}$ contains exactly one element of $[\{\mathbf{C}(\mathbf{X})\}_{i*} : i \in \mathcal{V}^c]$. Consequently, if $\mathbf{x}_{0i} = \mathbf{x}_{0j} = \mathbf{x}_{0k}^*$ for some $i, j \in \mathcal{V}^c$ and $k \in [K]$, then $\{\mathbf{C}(\mathbf{X})\}_{i*}, \{\mathbf{C}(\mathbf{X})\}_{j*} \in B_{\|\cdot\|_2}\{\mathbf{W}(\mathbf{X}_0, \mathbf{X})^{\mathrm{T}}\mathbf{x}_{0k}^*, \xi/2\}$, implying that $\{\mathbf{C}(\mathbf{X})\}_{i*} = \{\mathbf{C}(\mathbf{X})\}_{j*}$.

The above argument can be briefly stated as follows: $\mathbf{x}_{0i} = \mathbf{x}_{0j}$ if and only if $\{\mathbf{C}(\mathbf{X})\}_{i*} = \{\mathbf{C}(\mathbf{X})\}_{j*}$. This immediately implies that

$$\inf_{\sigma \in \mathcal{S}_K} d_H\{\sigma \circ \tau(\cdot; \mathbf{X}_0), \tau(\cdot; \mathbf{X})\} \leq \frac{16M_2^2}{\xi^2},$$

and the first assertion is proved by an application of Theorem 7.

To prove the second assertion, we need to apply the large deviation bound in Theorem 8. Note that for any $i, j \in [n]$ with $\mathbf{x}_{0i} \neq \mathbf{x}_{0j}$,

$$\begin{aligned}\|\mathbf{x}_{0i} - \mathbf{x}_{0j}\|_2^2 &= (\mathbf{e}_i - \mathbf{e}_j)^{\mathrm{T}}\mathbf{X}_0\mathbf{X}_0^{\mathrm{T}}(\mathbf{e}_i - \mathbf{e}_j) = \|\mathbf{S}_0^{1/2}\mathbf{U}_0^{\mathrm{T}}(\mathbf{e}_i - \mathbf{e}_j)\|_2^2 \leq \|\mathbf{X}_0\|_{\mathrm{F}}^2\|\mathbf{U}_0^{\mathrm{T}}(\mathbf{e}_i - \mathbf{e}_j)\|_2 \\ &\leq n(\mathbf{e}_i - \mathbf{e}_j)^{\mathrm{T}}\mathbf{U}_0\mathbf{U}_0^{\mathrm{T}}(\mathbf{e}_i - \mathbf{e}_j) = n\|(\mathbf{U}_0)_{i*} - (\mathbf{U}_0)_{j*}\|_2^2.\end{aligned}$$

By assumption, this implies $\|(\mathbf{U}_0)_{i*} - (\mathbf{U}_0)_{j*}\|_2 > \xi/\sqrt{n}$. Assume that the event $\left\{\|\widehat{\mathbf{U}} - \mathbf{U}_0\mathbf{W}_{\mathbf{U}}\|_{\mathrm{F}} \leq M'/\sqrt{n}\right\}$ occurs with respect to $\mathbb{P}_0$, where $M' = 8M_1\sqrt{2d}/\lambda_d(\boldsymbol{\Delta})$ is a constant. Similarly,

$$\|\mathbf{C}(\widehat{\mathbf{U}}) - \mathbf{U}_0\mathbf{W}_{\mathbf{U}}\|_{\mathrm{F}} \leq 2\|\widehat{\mathbf{U}} - \mathbf{U}_0\mathbf{W}_{\mathbf{U}}\| \leq \frac{2M'}{\sqrt{n}}.$$

An argument that is similar to that for the first assertion (up to a factor of $1/\sqrt{n}$) shows that

$$\inf_{\sigma\in\mathcal{S}_K} d_H\{\sigma\circ\tau(\cdot;\widehat{\mathbf{U}}), \tau(\cdot;\mathbf{U}_0)\} \leq \frac{16M'^2}{\xi^2}.$$

Namely,

$$\mathbb{P}_0\left[\inf_{\sigma\in\mathcal{S}_K} d_H\{\tau(\cdot;\widehat{\mathbf{U}}), \tau(\cdot;\mathbf{U}_0)\} > \frac{16M'^2}{\xi^2}\right] \leq 2\exp\left(-\frac{1}{4}M_1 d\sqrt{n}\right).$$

It follows immediately from Borel-Cantelli lemma that

$$\mathbb{P}_0\left[\inf_{\sigma\circ\sigma\in\mathcal{S}_K} d_H\{\sigma\circ\tau(\cdot;\widehat{\mathbf{U}}), \tau(\cdot;\mathbf{U}_0)\} \leq \frac{16M'^2}{\xi^2}\text{ almost always}\right] = 1.$$

The proof is completed by plugging-in M'. □

5.2.5 Proofs for Section 3.5

Lemma 19 *Let $\mathbf{E}\in\mathbb{R}^{n\times n}$ be a symmetric random matrix with $(A_{ij}: 1\leq i<j\leq n)$ being independent, and let $\mathbb{E}_0(\mathbf{E}) = \mathbf{0}_{n\times n}$. Assume that $\mathbf{E}$ are sub-Gaussian,* i.e., *there exists some constant $\tau > 0$, such that for all $\mathbf{A}\in\mathbb{R}^{n\times n}$ with $\|\mathbf{A}\|_{\mathrm{F}}^2 = 1$, for all $t > 0$, $\mathbb{P}_0\left(|\mathrm{Tr}\left(\mathbf{A}^{\mathrm{T}}\mathbf{A}\right)| > t\right) \leq 2\mathrm{e}^{-\tau t^2}$. For any $\mathbf{X}, \mathbf{X}_0\in\mathbb{R}^{n\times d}$ and for any $t > 0$,*

$$\mathbb{P}_0\left(\sup_{\mathbf{X}\in\mathbb{R}^{n\times d}}\left|\left\langle\mathbf{E}, \frac{\mathbf{X}\mathbf{X}^{\mathrm{T}} - \mathbf{X}_0\mathbf{X}_0^{\mathrm{T}}}{\|\mathbf{X}\mathbf{X}^{\mathrm{T}} - \mathbf{X}_0\mathbf{X}_0^{\mathrm{T}}\|_{\mathrm{F}}}\right\rangle_{\mathrm{F}}\right| > nt\right) \leq 6\exp\left(3nd - \frac{\tau n^2 t^2}{4}\right).$$

Proof. The proof is based on a popular discretization and covering technique (see, for example, [88]). First observe that

$$\left\{\sup_{\mathbf{X}\in\mathbb{R}^{n\times d}}\left|\left\langle\mathbf{E}, \frac{\mathbf{X}\mathbf{X}^{\mathrm{T}} - \mathbf{X}_0\mathbf{X}_0^{\mathrm{T}}}{\|\mathbf{X}\mathbf{X}^{\mathrm{T}} - \mathbf{X}_0\mathbf{X}_0^{\mathrm{T}}\|_{\mathrm{F}}}\right\rangle_{\mathrm{F}}\right| > nt\right\} \subset \left\{\sup_{\mathrm{rank}(\mathbf{B})\leq 2d, \|\mathbf{B}\|_{\mathrm{F}}=1}|\langle\mathbf{E},\mathbf{B}\rangle_{\mathrm{F}}| > nt\right\}.$$

Let $\mathcal{D}(1/6)$ be an 1/6-net of $\{\mathbf{S} = \mathrm{diag}(\sigma_1, \ldots, \sigma_{2d}) : \|\mathbf{S}\|_{\mathrm{F}} = 1, \sigma_i \geq 0\}$, and $\mathcal{O}(1/6)$ be an 1/6-net of $\{\mathbf{U} \in \mathbb{R}^{n\times 2d} : \|\mathbf{U}\|_{\mathrm{F}} = 1\}$. Clearly, $|\mathcal{O}(1/6)| \leq (18)^{nd}$ and $|\mathcal{D}(1/6)| \leq (18)^{2d}$ due to the covering number bounds of the Euclidean space [89]. For any $\mathbf{B}$ with $\mathrm{rank}(\mathbf{B}) \leq 2d$ and $\|\mathbf{B}\|_{\mathrm{F}} = 1$, let $\mathbf{B}$ admits singular value decomposition $\mathbf{B} = \mathbf{U}\mathbf{S}\mathbf{V}^{\mathrm{T}}$, where $\mathbf{U}, \mathbf{V} \in \mathbb{O}(n, 2d)$. Then there exists some $\widetilde{\mathbf{U}}, \widetilde{\mathbf{V}} \in \mathcal{O}(1/6)$ and $\widetilde{\mathbf{S}} \in \mathcal{D}(1/6)$, such that $\|\mathbf{U} - \widetilde{\mathbf{U}}\|_{\mathrm{F}} < 1/6$, $\|\mathbf{V} - \widetilde{\mathbf{V}}\|_{\mathrm{F}} < 1/6$, and $\|\mathbf{S} - \widetilde{\mathbf{S}}\|_{\mathrm{F}} < 1/6$. We proceed to derive

$$\begin{aligned}
|\langle \mathbf{E}, \mathbf{B}\rangle_{\mathrm{F}}| &= \left|\left\langle \mathbf{E}, \mathbf{U}\mathbf{S}\mathbf{V}^{\mathrm{T}}\right\rangle_{\mathrm{F}}\right| \\
&\leq \left|\left\langle \mathbf{E}, \mathbf{U}\mathbf{S}(\mathbf{V} - \widetilde{\mathbf{V}})^{\mathrm{T}}\right\rangle_{\mathrm{F}}\right| + \left|\left\langle \mathbf{E}, \mathbf{U}(\mathbf{S} - \widetilde{\mathbf{S}})\widetilde{\mathbf{V}}^{\mathrm{T}}\right\rangle_{\mathrm{F}}\right| + \left|\left\langle \mathbf{E}, (\mathbf{U} - \widetilde{\mathbf{U}})\widetilde{\mathbf{S}}\widetilde{\mathbf{V}}^{\mathrm{T}}\right\rangle_{\mathrm{F}}\right| \\
&\quad + \left|\left\langle \mathbf{E}, \widetilde{\mathbf{U}}\widetilde{\mathbf{S}}\widetilde{\mathbf{V}}^{\mathrm{T}}\right\rangle_{\mathrm{F}}\right| \\
&\leq \left|\left\langle \mathbf{E}, \frac{\mathbf{U}\mathbf{S}(\mathbf{V} - \widetilde{\mathbf{V}})^{\mathrm{T}}}{\|\mathbf{U}\mathbf{S}(\mathbf{V} - \widetilde{\mathbf{V}})^{\mathrm{T}}\|_{\mathrm{F}}}\right\rangle_{\mathrm{F}}\right| \|\mathbf{V} - \widetilde{\mathbf{V}}\|_{\mathrm{F}} \\
&\quad + \left|\left\langle \mathbf{E}, \frac{\mathbf{U}(\mathbf{S} - \widetilde{\mathbf{S}})\widetilde{\mathbf{V}}^{\mathrm{T}}}{\|\mathbf{U}(\mathbf{S} - \widetilde{\mathbf{S}})\widetilde{\mathbf{V}}^{\mathrm{T}}\|_{\mathrm{F}}}\right\rangle_{\mathrm{F}}\right| \|\mathbf{S} - \widetilde{\mathbf{S}}\|_{\mathrm{F}} \\
&\quad + \left|\left\langle \mathbf{E}, \frac{(\mathbf{U} - \widetilde{\mathbf{U}})\widetilde{\mathbf{S}}\widetilde{\mathbf{V}}^{\mathrm{T}}}{\|(\mathbf{U} - \widetilde{\mathbf{U}})\widetilde{\mathbf{S}}\widetilde{\mathbf{V}}^{\mathrm{T}}\|_{\mathrm{F}}}\right\rangle_{\mathrm{F}}\right| \|\mathbf{U} - \widetilde{\mathbf{U}}\|_{\mathrm{F}} \\
&\quad + \sup_{\widetilde{\mathbf{U}}, \widetilde{\mathbf{V}} \in \mathcal{O}(1/6), \mathbf{S} \in \mathcal{D}(1/6)} \left|\left\langle \mathbf{E}, \widetilde{\mathbf{U}}\widetilde{\mathbf{S}}\widetilde{\mathbf{V}}^{\mathrm{T}}\right\rangle_{\mathrm{F}}\right| \\
&\leq \frac{1}{2} \sup_{\mathrm{rank}(\mathbf{B}) \leq 2d, \|\mathbf{B}\|_{\mathrm{F}} = 1} |\langle \mathbf{E}, \mathbf{B}\rangle_{\mathrm{F}}| + \sup_{\widetilde{\mathbf{U}}, \widetilde{\mathbf{V}} \in \mathcal{O}(1/6), \mathbf{S} \in \mathcal{D}(1/6)} \left|\left\langle \mathbf{E}, \widetilde{\mathbf{U}}\widetilde{\mathbf{S}}\widetilde{\mathbf{V}}^{\mathrm{T}}\right\rangle_{\mathrm{F}}\right|.
\end{aligned}$$

Hence, we obtain, after taking the supremum with respect to $\{\mathbf{B} : \mathrm{rank}(\mathbf{B}) \leq 2d, \|\mathbf{B}\|_{\mathrm{F}} = 1\}$, that

$$\sup_{\mathrm{rank}(\mathbf{B}) \leq 2d, \|\mathbf{B}\|_{\mathrm{F}} = 1} |\langle \mathbf{E}, \mathbf{B}\rangle_{\mathrm{F}}| \leq 2 \sup_{\widetilde{\mathbf{U}}, \widetilde{\mathbf{V}} \in \mathcal{O}(1/6), \mathbf{S} \in \mathcal{D}(1/6)} \left|\left\langle \mathbf{E}, \widetilde{\mathbf{U}}\widetilde{\mathbf{S}}\widetilde{\mathbf{V}}^{\mathrm{T}}\right\rangle_{\mathrm{F}}\right|.$$

Therefore, by the union bound, we obtain

$$\begin{aligned}
&\mathbb{P}_0\left(\sup_{\mathbf{X} \in \mathbb{R}^{n\times d}} \left|\left\langle \mathbf{E}, \frac{\mathbf{X}\mathbf{X}^{\mathrm{T}} - \mathbf{X}_0\mathbf{X}_0^{\mathrm{T}}}{\|\mathbf{X}\mathbf{X}^{\mathrm{T}} - \mathbf{X}_0\mathbf{X}_0^{\mathrm{T}}\|_{\mathrm{F}}}\right\rangle_{\mathrm{F}}\right| > nt\right) \\
&\quad \leq \mathbb{P}\left(\sup_{\widetilde{\mathbf{U}}, \widetilde{\mathbf{V}} \in \mathcal{O}(1/6), \mathbf{S} \in \mathcal{D}(1/6)} \left|\left\langle \mathbf{E}, \widetilde{\mathbf{U}}\widetilde{\mathbf{S}}\widetilde{\mathbf{V}}^{\mathrm{T}}\right\rangle_{\mathrm{F}}\right| > \frac{nt}{2}\right) \\
&\quad \leq \sum_{\widetilde{\mathbf{U}} \in \mathcal{O}(1/6)} \sum_{\widetilde{\mathbf{V}} \in \mathcal{O}(1/6)} \sum_{\widetilde{\mathbf{S}} \in \mathcal{D}(1/6)} \mathbb{P}\left(\left|\left\langle \mathbf{E}, \widetilde{\mathbf{U}}\widetilde{\mathbf{S}}\widetilde{\mathbf{V}}^{\mathrm{T}}\right\rangle_{\mathrm{F}}\right| > \frac{nt}{2}\right)
\end{aligned}$$

$$\leq 6(18)^{nd}\exp(-\tau n t^2/4) \leq 6\exp\left(3nd - \frac{\tau n^2 t^2}{4}\right),$$

where we have invoke the condition that $\mathbf{E}$ has sub-Gaussian entries. The proof is thus completed. $\square$

Proof of Theorem 11. Let $\epsilon_n = \sqrt{d(\log n)/n}$. Simple algebra shows that

$$-\frac{1}{2}\|\mathbf{A} - \mathbf{X}\mathbf{X}^{\mathrm{T}}\|_{\mathrm{F}}^2 + \frac{1}{2}\|\mathbf{A} - \mathbf{X}_0\mathbf{X}_0^{\mathrm{T}}\|_{\mathrm{F}}^2 = -\frac{1}{2}\|\mathbf{X}\mathbf{X}^{\mathrm{T}} - \mathbf{X}_0\mathbf{X}_0^{\mathrm{T}}\|_{\mathrm{F}}^2 + \langle \mathbf{E}, \mathbf{X}\mathbf{X}^{\mathrm{T}} - \mathbf{X}_0\mathbf{X}_0^{\mathrm{T}}\rangle_{\mathrm{F}},$$

where $\mathbf{E} = \mathbf{A} - \mathbb{E}_0\mathbf{A}$. For any $\alpha > 0$, denote

$$\mathcal{E}_n(\alpha) = \left\{\sup_{\mathbf{X}\in\mathbb{R}^{n\times d}}\left|\left\langle \mathbf{E}, \frac{\mathbf{X}\mathbf{X}^{\mathrm{T}} - \mathbf{X}_0\mathbf{X}_0^{\mathrm{T}}}{\|\mathbf{X}\mathbf{X}^{\mathrm{T}} - \mathbf{X}_0\mathbf{X}_0^{\mathrm{T}}\|_{\mathrm{F}}}\right\rangle_{\mathrm{F}}\right| \leq \alpha n\epsilon_n\right\}.$$

Since $\mathbf{E}$ is sub-Gaussian, we can invoke Lemma 19 and obtain

$$\mathbb{P}_0\left\{\mathcal{E}_n(\alpha)^c\right\} \leq 6\exp\left(3nd - \frac{\tau n^2\alpha^2\epsilon_n^2}{4}\right) = 6\exp\left\{-\left(\frac{\alpha^2\tau}{4}\log n - 3\right)dn\right\}.$$

Denote $\mathcal{U}_n(\gamma) = \{\|\mathbf{X}\mathbf{X}^{\mathrm{T}} - \mathbf{X}_0\mathbf{X}_0^{\mathrm{T}}\|_{\mathrm{F}} \leq n\gamma\}$. Then over the event $\mathcal{E}_n(\alpha)$, the denominator D_n^G can be lower bounded as follows:

$$\begin{aligned}
D_n^G &\geq \int_{\mathcal{U}_n(\epsilon_n)}\exp\left(-\left|\left\langle \mathbf{E}, \frac{\mathbf{X}\mathbf{X}^{\mathrm{T}} - \mathbf{X}_0\mathbf{X}_0^{\mathrm{T}}}{\|\mathbf{X}\mathbf{X}^{\mathrm{T}} - \mathbf{X}_0\mathbf{X}_0^{\mathrm{T}}\|_{\mathrm{F}}}\right\rangle_{\mathrm{F}}\right|\|\mathbf{X}\mathbf{X}^{\mathrm{T}} - \mathbf{X}_0\mathbf{X}_0^{\mathrm{T}}\|_{\mathrm{F}}\right.\\
&\qquad\qquad\left.-\frac{1}{2}\|\mathbf{X}\mathbf{X}^{\mathrm{T}} - \mathbf{X}_0\mathbf{X}_0^{\mathrm{T}}\|_{\mathrm{F}}^2\right)\Pi(\mathrm{d}\mathbf{X})\\
&\geq \int_{\mathcal{U}_n(\epsilon_n)}\exp\left(-\frac{1}{2}\|\mathbf{X}\mathbf{X}^{\mathrm{T}} - \mathbf{X}_0\mathbf{X}_0^{\mathrm{T}}\|_{\mathrm{F}}^2 - \alpha n\epsilon_n\|\mathbf{X}\mathbf{X}^{\mathrm{T}} - \mathbf{X}_0\mathbf{X}_0^{\mathrm{T}}\|_{\mathrm{F}}\right)\Pi(\mathrm{d}\mathbf{X})\\
&\geq \int_{\mathcal{U}_n(\epsilon_n)}\exp\left(-\frac{1}{2}\|\mathbf{X}\mathbf{X}^{\mathrm{T}} - \mathbf{X}_0\mathbf{X}_0^{\mathrm{T}}\|_{\mathrm{F}}^2 - \frac{1}{2}\alpha^2 n^2\epsilon_n^2 - \frac{1}{2}\|\mathbf{X}\mathbf{X}^{\mathrm{T}} - \mathbf{X}_0\mathbf{X}_0^{\mathrm{T}}\|_{\mathrm{F}}^2\right)\Pi(\mathrm{d}\mathbf{X})\\
&\geq \Pi\{\mathcal{U}_n(\epsilon_n)\}\exp\left\{-\left(1 + \frac{\alpha^2}{2}\right)n^2\epsilon_n^2\right\}.
\end{aligned}$$

Observe that

$$\begin{aligned}
\|\mathbf{X}\mathbf{X}^{\mathrm{T}} - \mathbf{X}_0\mathbf{X}_0^{\mathrm{T}}\|_{\mathrm{F}} &\leq \|(\mathbf{X} - \mathbf{X}_0)(\mathbf{X} - \mathbf{X}_0)^{\mathrm{T}}\|_{\mathrm{F}} + \|\mathbf{X}_0(\mathbf{X} - \mathbf{X}_0)^{\mathrm{T}}\|_{\mathrm{F}} + \|(\mathbf{X} - \mathbf{X}_0)\mathbf{X}_0^{\mathrm{T}}\|_{\mathrm{F}}\\
&\leq \|\mathbf{X} - \mathbf{X}_0\|_{\mathrm{F}}^2 + 2\sqrt{n}\|\mathbf{X} - \mathbf{X}_0\|_{\mathrm{F}} = (2\sqrt{n} + \|\mathbf{X} - \mathbf{X}_0\|_{\mathrm{F}})\|\mathbf{X} - \mathbf{X}_0\|_{\mathrm{F}}.
\end{aligned}$$

It follows that

$$\left\{\mathbf{X} : \|\mathbf{X} - \mathbf{X}_0\|_{\mathrm{F}} \leq \frac{\sqrt{n}\epsilon_n}{3}\right\} \subset \{\|\mathbf{X}\mathbf{X}^{\mathrm{T}} - \mathbf{X}_0\mathbf{X}_0^{\mathrm{T}}\|_{\mathrm{F}} \leq n\epsilon_n\} = \mathcal{U}_n(\epsilon_n).$$

Note that the concentration of Gaussian distribution can be lower bounded by the Anderson's lemma:

$$\begin{aligned}
\Pi\{\mathcal{U}_n(\epsilon_n)\} &\geq \Pi\left(\mathbf{X} : \|\mathbf{X} - \mathbf{X}_0\|_{\mathrm{F}} \leq \frac{\sqrt{n}\epsilon_n}{3}\right) \\
&\geq \exp\left(-\frac{1}{2\sigma^2}\|\mathbf{X}_0\|_{\mathrm{F}}^2\right) \prod_{i=1}^{n}\prod_{k=1}^{d} \Pi\left(x_{jk}^2 \leq \frac{\epsilon_n^2}{9d}\right) \\
&= \exp\left(-\frac{1}{2\sigma^2}\|\mathbf{X}_0\|_{\mathrm{F}}^2\right) \prod_{i=1}^{n}\prod_{k=1}^{d} \left\{2\Phi\left(\frac{\epsilon_n}{3\sigma\sqrt{d}}\right) - 1\right\} \\
&\geq \exp\left\{-\left(\frac{1}{2\sigma^2} + d\right) n - nd\left|\log\frac{\epsilon_n}{3\sigma\sqrt{d}}\right|\right\} \\
&\geq \exp\left\{-\left(\frac{1}{2\sigma^2} + d + d|\log 3\sigma|\right) n - \frac{1}{2}nd\log n\right\}.
\end{aligned}$$

Hence, over the event $\mathcal{E}_n(\alpha)$, we obtain

$$\begin{aligned}
D_n^G &\geq \Pi\{\mathcal{U}_n(\epsilon_n)\}\exp\left\{-\left(1 + \frac{\alpha^2}{2}\right) n^2\epsilon_n^2\right\} \\
&\geq \exp\left\{-\left(\frac{\alpha^2+3}{2}\right) nd\log n - \left(\frac{1}{2\sigma^2} + d + d|\log 3\sigma|\right) n\right\}.
\end{aligned}$$

We proceed to bound $\mathbb{E}_0[\Pi\{\mathcal{U}_n^c(M\epsilon_n) \mid \mathbf{A}\}]$ as follows:

$$\begin{aligned}
&\mathbb{E}_0[\Pi\{\mathcal{U}_n^c(M\epsilon_n) \mid \mathbf{A}\}] \\
&\quad\leq \mathbb{E}_0\left[\mathbb{1}\{\mathbf{A} \in \mathcal{E}_n(\alpha)\}\frac{N_n^G\{\mathcal{U}_n^c(M\epsilon_n)\}}{D_n^G}\right] + \mathbb{P}_0\{\mathcal{E}_n^c(\alpha)\} \\
&\quad\leq \exp\left\{\left(\frac{\alpha^2+3}{2}\right) nd\log n + \left(\frac{1}{2\sigma^2} + d + d|\log 3\sigma|\right) n\right\} \\
&\qquad\times \int_{\mathcal{U}_n^c(M\epsilon_n)} \exp\left(-\frac{1}{2}\|\mathbf{X}\mathbf{X}^{\mathrm{T}} - \mathbf{X}_0\mathbf{X}_0^{\mathrm{T}}\|_{\mathrm{F}}^2 + \alpha n\epsilon_n\|\mathbf{X}\mathbf{X}^{\mathrm{T}} - \mathbf{X}_0\mathbf{X}_0^{\mathrm{T}}\|_{\mathrm{F}}\right)\Pi(\mathrm{d}\mathbf{X}) \\
&\qquad+ 6\exp\left\{-\left(\frac{\alpha^2\tau}{4}\log n - 3\right) dn\right\} \\
&\quad\leq \exp\left\{\left(\frac{\alpha^2+3}{2}\right) n\log n + \left(\frac{1}{2\sigma^2} + d + d|\log 3\sigma|\right) n\right\} \\
&\qquad\times \int_{\mathcal{U}_n^c(M\epsilon_n)} \exp\left(-\frac{1}{2}\|\mathbf{X}\mathbf{X}^{\mathrm{T}} - \mathbf{X}_0\mathbf{X}_0^{\mathrm{T}}\|_{\mathrm{F}}^2 + 2\alpha^2 n^2\epsilon_n^2 + \frac{1}{8}\|\mathbf{X}\mathbf{X}^{\mathrm{T}} - \mathbf{X}_0\mathbf{X}_0^{\mathrm{T}}\|_{\mathrm{F}}^2\right)\Pi(\mathrm{d}\mathbf{X}) \\
&\qquad+ 6\exp\left\{\left(\frac{\alpha^2\tau}{4}\log n - 3\right) nd\right\} \\
&\quad\leq \exp\left\{\left(\frac{1}{2\sigma^2} + d + d|\log 3\sigma|\right) n - \left(\frac{3}{8}M^2 - \frac{5\alpha^2+3}{2}\right) nd\log n\right\}
\end{aligned}$$

$$+ 6\exp\left\{-\left(\frac{\alpha^2\rho^2}{4}\log n - 3\right)nd\right\},$$

where the second inequality is due to Fubini's theorem, and the fourth inequality is due to the fact that $ab \leq 2a^2 + b^2/8$ for any $a, b > 0$. Hence, taking $\alpha = \sqrt{(M^2+6)/10}$, then for sufficiently large M, we see that

$$\begin{aligned}
\mathbb{E}_0[\Pi\{\mathcal{U}_n^c(M\epsilon_n) \mid \mathbf{A}\}] &\leq \exp\left(-\frac{1}{16}M^2 nd\log n\right) + 6\exp\left\{-\frac{(M^2+6)\rho^2}{80}nd\log n\right\}\\
&\leq 7\exp\left\{-\min\left(\frac{1}{16}, \frac{\tau}{80}\right)M^2 nd\log n\right\}.
\end{aligned}$$

Now set $\Xi_n = \{\mathbf{X} \in \mathbb{R}^{n\times d} : (1/n)\|\mathbf{X}\mathbf{X}^{\mathrm{T}} - \mathbf{X}_0\mathbf{X}_0^{\mathrm{T}}\|_{\mathrm{F}} \leq \sqrt{(Md\log n)/n}\}$. Following the same argument for deriving inequality (5.16), we see that there exists some constant $\widetilde{C} > 0$, such that for any $\mathbf{X} \in \Xi_n$ and sufficiently large n,

$$\inf_{\mathbf{W}\in\mathbb{O}(d)}\|\mathbf{X} - \mathbf{X}_0\mathbf{W}\|_{\mathrm{F}} \lesssim \sqrt{\frac{d}{n}}\|\mathbf{X}\mathbf{X}^{\mathrm{T}} - \mathbf{X}_0\mathbf{X}_0^{\mathrm{T}}\|_{\mathrm{F}}.$$

Therefore, there exists some constant $M' > 0$, such that for any $\mathbf{X} \in \Xi_n$,

$$\frac{1}{M'n}\inf_{\mathbf{W}\in\mathbb{O}(d)}\|\mathbf{X} - \mathbf{X}_0\mathbf{W}\|_{\mathrm{F}}^2 \leq \frac{1}{Mn^2}\|\mathbf{X}\mathbf{X}^{\mathrm{T}} - \mathbf{X}_0\mathbf{X}_0^{\mathrm{T}}\|_{\mathrm{F}}^2,$$

and hence,

$$\begin{aligned}
&\mathbb{E}_0\left\{\Pi_G\left(\frac{1}{n}\inf_{\mathbf{W}\in\mathbb{O}(d)}\|\mathbf{X} - \mathbf{X}_0\mathbf{W}\|_{\mathrm{F}}^2 > \frac{M'd\log n}{n}\,\Big|\,\mathbf{A}\right)\right\}\\
&\quad\leq \mathbb{E}_0\left\{\Pi_G\left(\mathbf{X}\in\Xi_n : \frac{1}{n}\inf_{\mathbf{W}\in\mathbb{O}(d)}\|\mathbf{X} - \mathbf{X}_0\mathbf{W}\|_{\mathrm{F}}^2 > \frac{M'd\log n}{n}\,\Big|\,\mathbf{A}\right)\right\}\\
&\qquad + \mathbb{E}_0\left\{\Pi_G\left(\mathbf{X}\in\Xi_n^c \mid \mathbf{A}\right)\right\}\\
&\quad\leq \mathbb{E}_0\left\{\Pi_G\left(\frac{1}{n^2}\|\mathbf{X}\mathbf{X}^{\mathrm{T}} - \mathbf{X}_0\mathbf{X}_0^{\mathrm{T}}\|_{\mathrm{F}}^2 > \frac{Md\log n}{n}\,\Big|\,\mathbf{A}\right)\right\} + \mathbb{E}_0\left\{\Pi_G\left(\mathbf{X}\in\Xi_n^c \mid \mathbf{A}\right)\right\}\\
&\quad\leq 2\mathbb{E}_0\left\{\Pi_G\left(\mathbf{X}\in\Xi_n^c \mid \mathbf{A}\right)\right\} \leq 14\exp\left\{-\min\left(\frac{1}{16}, \frac{\tau}{80}\right)M^2 nd\log n\right\},
\end{aligned}$$

completing the proof. $\square$

5.2.6 Proofs for Section 3.6

We now prove the above results by first generalizing Lemma 15. The proof is similar to that of Lemma 15 and is presented for completeness.

Lemma 20 *Let* $\mathbf{A} \sim \mathrm{RDPG}(\mathbf{X}_0, \widetilde{\mathbf{X}}_0; \vartheta_n)$ *for some* $\mathbf{X}_0, \widetilde{\mathbf{X}}_0 \in \mathcal{X}^n$ *and some sequence* $(\vartheta_n)_{n=1}^{\infty}$, $\vartheta_n \in (0, 1]$ *for all* n. *Assume that* $\delta \leq \min_{i,j} \mathbf{x}_{0i}^{\mathrm{T}} \widetilde{\mathbf{x}}_{0j} \leq \max_{i,j} \mathbf{x}_{0i}^{\mathrm{T}} \widetilde{\mathbf{x}}_{0j} \leq 1 - \delta$ *for some constant* $\delta \in (0, 1/2)$ *independent of* n, *and that* $\pi_{\mathbf{x}}$ *is bounded away from* 0 *and* ∞. *Then for any constants* $\beta, \gamma > 0$, *and for sufficiently large* n *and sufficiently small* $\epsilon > 0$,

$$
\begin{aligned}
&\mathbb{P}_0 \left[D_n \leq \exp \left\{ 2(c_\pi + d \log \beta) n - \left(\frac{16 \vartheta_n \beta^2}{\delta^2} + \gamma \right) n^2 \epsilon^2 - nd \left(\log \frac{1}{\epsilon} \right) \right\} \right] \\
&\quad \leq \exp \left(- \frac{\gamma^2 \delta^4 n^2 \epsilon^2}{128 \beta^2} \right)
\end{aligned}
$$

for some constant c_π *independent of* n *and* d, *where* D_n *takes the form*

$$
\iint_{\mathcal{X}^n \times \mathcal{X}^n} \prod_{i,j \in [n]} \frac{p(A_{ij} \mid \mathbf{X}, \widetilde{\mathbf{X}})}{p(A_{ij} \mid \mathbf{X}_0, \widetilde{\mathbf{X}}_0)} \Pi(\mathrm{d}\mathbf{X} \mathrm{d}\widetilde{\mathbf{X}}).
$$

Proof. For any constant $\beta > 0$, set

$$
\mathcal{E}_n = \left\{ (\mathbf{X}, \widetilde{\mathbf{X}}) : \|\mathbf{X} - \mathbf{X}_0\|_{2 \to \infty} < \beta \epsilon, \|\widetilde{\mathbf{X}} - \widetilde{\mathbf{X}}_0\|_{2 \to \infty} < \beta \epsilon \right\}.
$$

Denote $P_{ij} = \mathbf{x}_i^{\mathrm{T}} \widetilde{\mathbf{x}}_j$ and $P_{0ij} = \mathbf{x}_{0i}^{\mathrm{T}} \widetilde{\mathbf{x}}_{0j}$, $i, j \in [n]$. Similar to the proof of Lemma 15, write

$$
\begin{aligned}
D_n &\geq \Pi(\mathcal{E}_n) \iint_{\mathcal{E}_n} \prod_{i,j \in [n]} \frac{p(A_{ij} \mid \mathbf{X}, \widetilde{\mathbf{X}})}{p(A_{ij} \mid \mathbf{X}_0, \widetilde{\mathbf{X}}_0)} \Pi(\mathrm{d}\mathbf{X} \mathrm{d}\widetilde{\mathbf{X}} \mid \mathcal{E}_n) \\
&= \Pi(\mathcal{E}_n) \iint_{\mathcal{E}_n} \exp \left[\sum_{i=1}^n \sum_{j=1}^n \left\{ -A_{ij} \log \left(\frac{\mathbf{P}_{0ij}}{\mathbf{P}_{ij}} \right) - (1 - A_{ij}) \log \left(\frac{1 - \vartheta_n \mathbf{P}_{0ij}}{1 - \vartheta_n \mathbf{P}_{ij}} \right) \right\} \right] \\
&\quad \times \Pi(\mathrm{d}\mathbf{X} \mathrm{d}\widetilde{\mathbf{X}} \mid \mathcal{E}_n) \\
&\geq \Pi(\mathcal{E}_n) \exp \left\{ - \sum_{i=1}^n \sum_{j=1}^n (A_{ij} - \vartheta_n \mathbf{P}_{0ij}) \iint_{\mathcal{E}_n} \frac{(\mathbf{P}_{ij} - \mathbf{P}_{0ij})}{\mathbf{P}_{ij}(1 - \vartheta_n \mathbf{P}_{ij})} \Pi(\mathrm{d}\mathbf{X} \mathrm{d}\widetilde{\mathbf{X}} \mid \mathcal{E}_n) \right\} \\
&\quad \times \exp \left\{ -\vartheta_n \sum_{i=1}^n \sum_{j=1}^n \iint_{\mathcal{E}_n} \frac{(\mathbf{P}_{0ij} - \mathbf{P}_{ij})^2}{\mathbf{P}_{ij}(1 - \vartheta_n \mathbf{P}_{ij})} \Pi(\mathrm{d}\mathbf{X} \mathrm{d}\widetilde{\mathbf{X}} \mid \mathcal{E}_n) \right\}.
\end{aligned}
$$

Since for any $(\mathbf{X}, \widetilde{\mathbf{X}}) \in \mathcal{E}_n$, we have, for any $i, j \in [n]$,

$$
\begin{aligned}
|\mathbf{P}_{ij} - \mathbf{P}_{0ij}| &\leq \|\widetilde{\mathbf{X}} - \widetilde{\mathbf{X}}_0\|_{2 \to \infty} + \|\mathbf{X} - \mathbf{X}_0\|_{2 \to \infty} \leq \frac{\delta}{2}, \\
\mathbf{P}_{ij}(1 - \vartheta_n \mathbf{P}_{ij}) &\geq (\mathbf{x}_{0i}^{\mathrm{T}} \widetilde{\mathbf{x}}_{0j} - |\mathbf{P}_{ij} - \mathbf{P}_{0ij}|)(1 - \mathbf{x}_{0i}^{\mathrm{T}} \widetilde{\mathbf{x}}_{0j} - |\mathbf{P}_{ij} - \mathbf{P}_{0ij}|) \geq \frac{\delta^2}{4},
\end{aligned}
$$

$$
\begin{aligned}
\left|\frac{\mathbf{P}_{ij}-\mathbf{P}_{0ij}}{\mathbf{P}_{ij}(1-\vartheta_n\mathbf{P}_{ij})}\right| &\leq \max_{i,j\in[n]}\frac{|(\mathbf{x}_i-\mathbf{x}_{0i})^{\mathrm{T}}\widetilde{\mathbf{x}}_j|+|\mathbf{x}_{0i}^{\mathrm{T}}(\widetilde{\mathbf{x}}_j-\widetilde{\mathbf{x}}_{0j})|}{\mathbf{P}_{ij}(1-\vartheta_n\mathbf{P}_{ij})}\leq\frac{8\beta\epsilon}{\delta^2},\\
\frac{(\mathbf{P}_{ij}-\mathbf{P}_{0ij})^2}{\mathbf{P}_{ij}(1-\vartheta_n\mathbf{P}_{ij})} &\leq \max_{i,j\in[n]}\frac{2|(\mathbf{x}_i-\mathbf{x}_{0i})^{\mathrm{T}}\widetilde{\mathbf{x}}_j|^2+2|\mathbf{x}_{0i}^{\mathrm{T}}(\widetilde{\mathbf{x}}_j-\widetilde{\mathbf{x}}_{0j})|^2}{\mathbf{P}_{ij}(1-\vartheta_n\mathbf{P}_{ij})}\leq\frac{16\beta^2\epsilon^2}{\delta^2},
\end{aligned}
$$

implying that

$$
\begin{aligned}
\left|\iint_{\mathcal{E}_n}\frac{(\mathbf{P}_{ij}-\mathbf{P}_{0ij})}{\mathbf{P}_{ij}(1-\vartheta_n\mathbf{P}_{ij})}\Pi(\mathrm{d}\mathbf{X}\mathrm{d}\widetilde{\mathbf{X}}\mid\mathcal{E}_n)\right| &\leq\frac{8\beta\epsilon}{\delta^2},\\
\iint_{\mathcal{E}_n}\frac{(\mathbf{P}_{ij}-\mathbf{P}_{0ij})^2}{\mathbf{P}_{ij}(1-\vartheta_n\mathbf{P}_{ij})}\Pi(\mathrm{d}\mathbf{X}\mathrm{d}\widetilde{\mathbf{X}}\mid\mathcal{E}_n) &\leq\frac{16\beta^2\epsilon^2}{\delta^2},
\end{aligned}
$$

it follows that for any $\gamma>0$,

$$
\begin{aligned}
&D_n\leq\Pi(\mathcal{E}_n)\exp\left\{-n^2\left(\frac{16\vartheta_n\beta^2\epsilon^2}{\delta^2}\right)-\gamma n^2\epsilon^2\right\}\\
&\qquad\Longrightarrow\sum_{i=1}^n\sum_{j=i}^n(A_{ij}-\vartheta_n\mathbf{P}_{0ij})\iint_{\mathcal{E}_n}\frac{(\mathbf{P}_{ij}-\mathbf{P}_{0ij})}{\mathbf{P}_{ij}(1-\vartheta_n\mathbf{P}_{ij})}\Pi(\mathrm{d}\mathbf{X}\mathrm{d}\widetilde{\mathbf{X}}\mid\mathcal{E}_n)>\gamma n^2\epsilon^2.
\end{aligned}
$$

Hence, by Hoeffding's inequality,

$$
\begin{aligned}
&\mathbb{P}_0\left[D_n\leq\Pi(\mathcal{E}_n)\exp\left\{-\frac{16\vartheta_n\beta^2n^2\epsilon^2}{\delta^2}-\gamma n^2\epsilon^2\right\}\right]\\
&\quad\leq\mathbb{P}_0\left\{\sum_{i=1}^n\sum_{j=1}^n(A_{ij}-\vartheta_n\mathbf{P}_{0ij})\iint_{\mathcal{E}_n}\frac{(\mathbf{P}_{ij}-\mathbf{P}_{0ij})}{\mathbf{P}_{ij}(1-\vartheta_n\mathbf{P}_{ij})}\Pi(\mathrm{d}\mathbf{X}\mathrm{d}\widetilde{\mathbf{X}}\mid\mathcal{E}_n)>\gamma n^2\epsilon^2\right\}\\
&\quad\leq\exp\left\{-\frac{2\gamma^2n^4\epsilon^4}{n^2}\left(\frac{\delta^2}{16\beta\epsilon}\right)^2\right\}=\exp\left(-\frac{\gamma^2\delta^4n^2\epsilon^2}{128\beta^2}\right).
\end{aligned}
$$

It suffices to provide an exponential lower bound for $\Pi(\mathcal{E}_n)$. This can be easily obtained using the fact that $\pi_{\mathbf{x}}(\mathbf{x}_i)\mathrm{vol}(B_1^d)\geq\exp(c_\pi)>0$ for some constant c_π: for sufficiently small ϵ, $\{\|\mathbf{x}_i-\mathbf{x}_{0i}\|_2<\beta\epsilon\}\subset\mathcal{X}$, $\{\|\widetilde{\mathbf{x}}_i-\widetilde{\mathbf{x}}_{0i}\|_2<\beta\epsilon\}\subset\mathcal{X}$, and thus,

$$
\begin{aligned}
\Pi(\mathcal{E}_n)&=\left\{\prod_{i=1}^n\int_{\{\|\mathbf{x}_i-\mathbf{x}_{0i}\|_2<\beta\epsilon\}}\pi_{\mathbf{x}}(\mathbf{x}_i)\mathrm{d}\mathbf{x}_i\right\}\left\{\prod_{j=1}^n\int_{\{\|\widetilde{\mathbf{x}}_j-\widetilde{\mathbf{x}}_{0j}\|_2<\beta\epsilon\}}\pi_{\mathbf{x}}(\widetilde{\mathbf{x}}_j)\mathrm{d}\widetilde{\mathbf{x}}_j\right\}\\
&\geq\exp\left(2nc_\pi\right)\left(\beta\epsilon\right)^{2nd}=\exp\left\{n(2c_\pi+2d\log\beta)-2nd\left(\log\frac{1}{\epsilon}\right)\right\}.
\end{aligned}
$$

Namely, for any $\beta,\gamma>0$, we obtain the following conclusion,

$$
\mathbb{P}_0\left[D_n\leq\exp\left\{2n\left(c_\pi+d\log\beta\right)-\left(\frac{16\vartheta_n\beta^2}{\delta^2}+\gamma\right)n^2\epsilon^2-nd\left(\log\frac{1}{\epsilon}\right)\right\}\right]
$$

$$\leq \exp\left(-\frac{\gamma^2\delta^4 n^2\epsilon^2}{128\beta^2}\right),$$

where c_π is some constant depending independent of d and n. The proof is thus completed. $\square$

Proof of Theorem 12. Take $\epsilon_n = \vartheta_n^{-1}\sqrt{n^{-1}d\log n}$. Let $\beta, \gamma > 0$ be constants to be determined later. Denote the event Ξ_n to be all $\mathbf{A}$ such that

$$D_n > \exp\left\{2(c_\pi + d\log\beta)n - \left(\frac{16\vartheta_n\beta^2}{\delta^2} + \gamma\right)n^2\vartheta_n^2\epsilon_n^2 - nd\left(\log\frac{1}{\vartheta_n\epsilon_n}\right)\right\}.$$

Consider the pseudo-metric $\rho : \mathcal{X}^n \times \mathcal{X}^n \to [0, \infty)$ defined by

$$\rho\{(\mathbf{X}_1, \widetilde{\mathbf{X}}_1), (\mathbf{X}_2, \widetilde{\mathbf{X}}_2)\} = \frac{1}{n}\|\mathbf{X}_1\widetilde{\mathbf{X}}_1^{\mathrm{T}} - \mathbf{X}_2\widetilde{\mathbf{X}}_2^{\mathrm{T}}\|_{\mathrm{F}}.$$

Let $\{(\mathbf{X}_1, \widetilde{\mathbf{X}}_1), \ldots, (\mathbf{X}_s, \widetilde{\mathbf{X}}_s)\}$ be an $\epsilon_n/2$-net of

$$[(\mathbf{X}, \widetilde{\mathbf{X}}) \in \mathcal{X}^n \times \mathcal{X}^n : \rho\{(\mathbf{X}, \widetilde{\mathbf{X}}), (\mathbf{X}_0, \widetilde{\mathbf{X}}_0)\} > M\epsilon_n]$$

with regard to the pseudo-metric ρ. Clearly,

$$\begin{aligned}
\rho\{(\mathbf{X}_1, \widetilde{\mathbf{X}}_1), (\mathbf{X}_2, \widetilde{\mathbf{X}}_2)\} &\leq \frac{1}{n}\left\{\|\mathbf{X}_1(\widetilde{\mathbf{X}}_1 - \widetilde{\mathbf{X}}_2)^{\mathrm{T}}\|_{\mathrm{F}} + \|(\mathbf{X}_1 - \mathbf{X}_2)\widetilde{\mathbf{X}}_2^{\mathrm{T}}\|_{\mathrm{F}}\right\}\\
&\leq \frac{1}{\sqrt{n}}\|\mathbf{X}_1 - \mathbf{X}_2\|_{\mathrm{F}} + \frac{1}{\sqrt{n}}\|\widetilde{\mathbf{X}}_1 - \widetilde{\mathbf{X}}_2\|_{\mathrm{F}},
\end{aligned}$$

implying that

$$\begin{aligned}
s &\leq \mathcal{N}\left(\frac{\epsilon_n}{2}, \mathcal{X}^n \times \mathcal{X}^n, \rho\right) \leq \mathcal{N}\left(\frac{\epsilon_n}{2}, \mathcal{X}^n, \frac{1}{\sqrt{n}}\|\cdot\|_{\mathrm{F}}\right) \times \mathcal{N}\left(\frac{\epsilon_n}{2}, \mathcal{X}^n, \frac{1}{\sqrt{n}}\|\cdot\|_{\mathrm{F}}\right)\\
&\leq \left\{\mathcal{N}\left[\frac{\sqrt{n}\epsilon_n}{2}, \{\mathbf{X} \in \mathbb{R}^{n\times d} : \|\mathbf{X}\|_{\mathrm{F}} \leq \sqrt{n}\}, \|\cdot\|_{\mathrm{F}}\right]\right\}^2 \leq \left(\frac{12}{\epsilon_n}\right)^{2nd}.
\end{aligned}$$

For each $r = 1, \ldots, s$, it can be seen that $(\mathbf{X}, \widetilde{\mathbf{X}}) \in B_\rho\{(\mathbf{X}_r, \widetilde{\mathbf{X}}_r), \epsilon_n/2\}$ implies that

$$\rho\{(\mathbf{X}, \widetilde{\mathbf{X}}), (\mathbf{X}_r, \widetilde{\mathbf{X}}_r)\} < \frac{\epsilon_n}{2} \leq \frac{M\epsilon_n}{2} \leq \frac{1}{2}\rho\{(\mathbf{X}_r, \widetilde{\mathbf{X}}_r), (\mathbf{X}_0, \widetilde{\mathbf{X}}_0)\}.$$

This allows us to invoke Lemma 17 to construct test functions ϕ_{rn}, $r \in [s]$, such that

$$\mathbb{E}_0\phi_{rn} \leq \exp\left[-C_1 n^2\vartheta_n^2\rho^2\{(\mathbf{X}_r, \widetilde{\mathbf{X}}_r), (\mathbf{X}_0, \widetilde{\mathbf{X}}_0)\}\right]$$

$$\le \exp\left(-Kn^2\vartheta_n^2 M^2\epsilon_n^2\right) = \exp(-KM^2 nd\log n),$$

$$\sup_{(\mathbf{X},\widetilde{\mathbf{X}})\in B_\rho\{(\mathbf{X}_r,\widetilde{\mathbf{X}}_r),\epsilon_n/2\}} \mathbb{E}_{(\mathbf{X},\widetilde{\mathbf{X}})}\left(1-\phi_{rn}\right) \le \exp\left\{-C_2 n^2\vartheta_n^2\rho^2(\mathbf{X}_r,\mathbf{X}_0)\right\}$$
$$\le \exp\left(-Kn^2\vartheta_n^2 M^2\epsilon_n^2\right) = \exp(-KM^2 nd\log n)$$

for some constant $K = \min\{C_1, C_2\}$. Taking $\phi_n = \max_{r\in[s]}\phi_{rn}$ yields the following bounds for the type I and type II error probabilities:

$$\begin{aligned}
\mathbb{E}_0\phi_n &= \mathbb{E}_0\left(\max_{r\in[s]}\phi_{rn}\right) \le \sum_{r=1}^{s}\mathbb{E}_0(\phi_{rn})\\
&\le \exp\left\{2nd\log 12 + 2nd\left(\log\frac{1}{\epsilon_n}\right) - KM^2 nd\log n\right\}\\
&\le \exp\left\{-\left(KM^2-6\right)nd\log n\right\},
\end{aligned}$$

and

$$\begin{aligned}
&\sup_{(\mathbf{X},\widetilde{\mathbf{X}}):\rho\{(\mathbf{X},\widetilde{\mathbf{X}}),(\mathbf{X}_0,\widetilde{\mathbf{X}}_0)\}>M\epsilon_n} \mathbb{E}_{(\mathbf{X},\widetilde{\mathbf{X}})}(1-\phi_n)\\
&\le \max_{r\in[s]}\sup_{(\mathbf{X},\widetilde{\mathbf{X}})\in B_\rho\{(\mathbf{X}_r,\widetilde{\mathbf{X}}_r),\epsilon_n/2\}}\mathbb{E}_{(\mathbf{X},\widetilde{\mathbf{X}})}\left(1-\max_{r\in[s]}\phi_{rn}\right)\\
&\le \max_{r\in[s]}\sup_{(\mathbf{X},\widetilde{\mathbf{X}})\in B_\rho\{(\mathbf{X}_r,\widetilde{\mathbf{X}}_r),\epsilon_n/2\}}\mathbb{E}_{(\mathbf{X},\widetilde{\mathbf{X}})}\left(1-\phi_{rn}\right)\\
&\le \exp(-KM^2 nd\log n).
\end{aligned}$$

We are now in a position to provide an exponential upper bound for $\mathbb{E}_0\left[\Pi\{\rho(\mathbf{X},\mathbf{X}_0) > M\epsilon_n \mid \mathbf{A}\}\right]$:

$$\begin{aligned}
&\mathbb{E}_0\left\{\Pi[\rho\{(\mathbf{X},\widetilde{\mathbf{X}}),(\mathbf{X}_0,\widetilde{\mathbf{X}}_0)\} > M\epsilon_n \mid \mathbf{A}]\right\}\\
&\quad\le \mathbb{E}_0\left\{\frac{N_n[(\mathbf{X},\widetilde{\mathbf{X}}):\rho\{(\mathbf{X},\widetilde{\mathbf{X}}),(\mathbf{X}_0,\widetilde{\mathbf{X}}_0)\} > M\epsilon_n]}{D_n}\mathbb{1}(\mathbf{A}\in\Xi_n)(1-\phi_n)\right\}\\
&\qquad + \mathbb{E}_0(\phi_n) + \mathbb{P}_0(\Xi_n^c)\\
&\quad\le \max_{\mathbf{A}\in\Xi_n}\left(\frac{1}{D_n}\right)\mathbb{E}_0\left\{(1-\phi_n)\iint_{\{(\mathbf{X},\widetilde{\mathbf{X}}):\rho\{(\mathbf{X},\widetilde{\mathbf{X}}),(\mathbf{X}_0,\widetilde{\mathbf{X}}_0)\}>M\epsilon_n\}}\frac{p(\mathbf{A}\mid\mathbf{X},\widetilde{\mathbf{X}})}{p(\mathbf{A}\mid\mathbf{X}_0,\widetilde{\mathbf{X}}_0)}\Pi(\mathrm{d}\mathbf{X}\mathrm{d}\widetilde{\mathbf{X}})\right\}\\
&\qquad + \mathbb{E}_0(\phi_n) + \mathbb{P}_0(\Xi_n^c).
\end{aligned}$$

By Fubini's theorem and inequality (5.18), the expected value appearing in the first term of the right-hand side of the above display can be further upper bounded:

$$\mathbb{E}_0\left\{(1-\phi_n)\iint_{\{(\mathbf{X},\widetilde{\mathbf{X}}):\rho\{(\mathbf{X},\widetilde{\mathbf{X}}),(\mathbf{X}_0,\widetilde{\mathbf{X}}_0)\}>M\epsilon_n\}}\frac{p(\mathbf{A}\mid\mathbf{X},\widetilde{\mathbf{X}})}{p(\mathbf{A}\mid\mathbf{X}_0,\widetilde{\mathbf{X}}_0)}\Pi(\mathrm{d}\mathbf{X}\mathrm{d}\widetilde{\mathbf{X}})\right\}$$

$$
\begin{aligned}
&= \iint_{\{(\mathbf{X},\widetilde{\mathbf{X}}):\rho\{(\mathbf{X},\widetilde{\mathbf{X}}),(\mathbf{X}_0,\widetilde{\mathbf{X}}_0)\}>M\epsilon_n\}} \mathbb{E}_0\left\{(1-\phi_n)\frac{p(\mathbf{A}\mid \mathbf{X},\widetilde{\mathbf{X}})}{p(\mathbf{A}\mid \mathbf{X}_0,\widetilde{\mathbf{X}}_0)}\right\}\Pi(\mathrm{d}\mathbf{X}\mathrm{d}\widetilde{\mathbf{X}})\\
&\le \iint \sup_{\{(\mathbf{X},\widetilde{\mathbf{X}}):\rho\{(\mathbf{X},\widetilde{\mathbf{X}}),(\mathbf{X}_0,\widetilde{\mathbf{X}}_0)\}>M\epsilon_n\}} \mathbb{E}_{(\mathbf{X},\widetilde{\mathbf{X}})}\left\{(1-\phi_n)\right\}\Pi(\mathrm{d}\mathbf{X}\mathrm{d}\widetilde{\mathbf{X}})\\
&\le \exp(-KM^2 nd\log n).
\end{aligned}
$$

Hence, invoking Lemma 15 and inequality (5.17) and setting $\beta=\delta^2$, $\gamma=8$, we have, for some constant $c(\delta)$ depending only on δ, that

$$
\begin{aligned}
&\mathbb{E}_0\left\{\Pi[\rho\{(\mathbf{X},\widetilde{\mathbf{X}}),(\mathbf{X}_0,\widetilde{\mathbf{X}}_0)\}>M\epsilon_n\mid \mathbf{A}]\right\}\\
&\quad\le \exp\left\{-2(c_\pi+d\log\delta^2)n+(16\delta^2+10)nd\log n-KM^2nd\log n\right\}\\
&\qquad+\exp\left\{-\left(KM^2-6\right)nd\log n\right\}+\exp\left(-\frac{nd\log n}{2}\right)\\
&\quad\le 2\exp\left[-\left\{KM^2-c(\delta)\right\}nd\log n\right]+\exp\left(-\frac{nd\log n}{2}\right).
\end{aligned}
$$

Taking M sufficiently large such that $KM^2-c(\delta)>1/2$ completes the proof. $\square$

We now proceed to prove Theorem 13 for the directe stochastic block model. In prepration for doing so, we need several technical lemmas.

Lemma 21 *Let $\mathbf{\Delta}_0=\sum_{k=1}^K w_k(\mathbf{x}_{0k}^*)(\mathbf{x}_{0k}^*)^{\mathrm{T}}$ and $\widetilde{\mathbf{\Delta}}_0=\sum_{k=1}^K w_k(\widetilde{\mathbf{x}}_{0k}^*)(\widetilde{\mathbf{x}}_{0k}^*)^{\mathrm{T}}$. Then for sufficiently large n, $\sigma_d(\mathbf{X}_0\widetilde{\mathbf{X}}_0^{\mathrm{T}})\ge n\lambda_d^{1/2}(\mathbf{\Delta}_0)\lambda_d^{1/2}(\widetilde{\mathbf{\Delta}}_0)/2$.*

Proof. Note that the matrices $\mathbf{X}_0\widetilde{\mathbf{X}}_0^{\mathrm{T}}\widetilde{\mathbf{X}}_0\mathbf{X}_0^{\mathrm{T}}$ and $\widetilde{\mathbf{X}}_0^{\mathrm{T}}\widetilde{\mathbf{X}}_0\mathbf{X}_0^{\mathrm{T}}\mathbf{X}_0$ have the same set of eigenvalues. Furthermore, for sufficiently large n, we see that $\lambda_d(\mathbf{X}_0^{\mathrm{T}}\mathbf{X}_0)\ge n\lambda_d(\mathbf{\Delta}_0)/2$, and similarly $\lambda_d(\widetilde{\mathbf{X}}_0^{\mathrm{T}}\widetilde{\mathbf{X}}_0)\ge n\lambda_d(\widetilde{\mathbf{\Delta}}_0)/2$, since $(1/n)\mathbf{X}_0^{\mathrm{T}}\mathbf{X}_0\to\mathbf{\Delta}_0$ and $(1/n)\widetilde{\mathbf{X}}_0^{\mathrm{T}}\widetilde{\mathbf{X}}_0\to\widetilde{\mathbf{\Delta}}_0$ as $n\to\infty$. Using the fact that $\lambda_d(\widetilde{\mathbf{X}}_0^{\mathrm{T}}\widetilde{\mathbf{X}}_0\mathbf{X}_0^{\mathrm{T}}\mathbf{X}_0)\ge\lambda_d(\widetilde{\mathbf{X}}_0^{\mathrm{T}}\widetilde{\mathbf{X}}_0)\lambda_d(\mathbf{X}_0^{\mathrm{T}}\mathbf{X}_0)$ ([90], Corollary 11), we obtain that $\sigma_d(\mathbf{X}_0\widetilde{\mathbf{X}}_0^{\mathrm{T}})^2=\lambda_d(\mathbf{X}_0\widetilde{\mathbf{X}}_0^{\mathrm{T}}\widetilde{\mathbf{X}}_0\mathbf{X}_0^{\mathrm{T}})=\lambda_d(\widetilde{\mathbf{X}}_0^{\mathrm{T}}\widetilde{\mathbf{X}}_0\mathbf{X}_0^{\mathrm{T}}\mathbf{X}_0)\ge\lambda_d(\widetilde{\mathbf{X}}_0^{\mathrm{T}}\widetilde{\mathbf{X}}_0)\lambda_d(\mathbf{X}_0^{\mathrm{T}}\mathbf{X}_0)\ge n^2\lambda_d(\mathbf{\Delta}_0)\lambda_d(\widetilde{\mathbf{\Delta}}_0)/4$. The proof is thus completed.

$\square$

The following Lemma is an adaptation of Lemma 3 in [33], and the proof follows the exact same lines of that in [33].

Lemma 22 *Let* $\mathbf{U}_0 = [\mathbf{u}_{01}, \ldots, \mathbf{u}_{0n}]^{\mathrm{T}}$*. Then* $\|\mathbf{u}_{0i} - \mathbf{u}_{0j}\|_2 \geq \xi\lambda_d^{1/4}(\mathbf{\Delta}_0)\lambda_d^{1/4}(\widetilde{\mathbf{\Delta}}_0)(2n)^{-1/2}$ *if* $i, j \in [n]$ *are such that* $\mathbf{x}_{0i} \neq \mathbf{x}_{0j}$*.*

Lemma 23 *Assume that* $(1/n)\|\mathbf{X}\widetilde{\mathbf{X}}^{\mathrm{T}} - \mathbf{X}_0\widetilde{\mathbf{X}}_0^{\mathrm{T}}\|_{\mathrm{F}} \leq M\vartheta_n^{-1}\sqrt{n^{-1}d\log n}$*, and* $\mathbf{X}, \widetilde{\mathbf{X}} \in \mathcal{X}^n$*. Then there exists an orthogonal matrix* $\widetilde{\mathbf{W}}$ *such that* $\|\mathbf{U} - \mathbf{U}_0\widetilde{\mathbf{W}}\|_{\mathrm{F}} \lesssim \vartheta_n^{-1}\sqrt{n^{-1}\log n}$*.*

Proof. Denote $\mathbf{P} = \mathbf{X}\widetilde{\mathbf{X}}^{\mathrm{T}}$ and $\mathbf{P}_0 = \mathbf{X}_0\widetilde{\mathbf{X}}_0^{\mathrm{T}}$. Observe that

$$\begin{aligned}\|\mathbf{P}\mathbf{P}^{\mathrm{T}} - \mathbf{P}_0\mathbf{P}_0^{\mathrm{T}}\|_{\mathrm{F}} &\leq \|(\mathbf{P} - \mathbf{P}_0)\mathbf{P}^{\mathrm{T}}\|_{\mathrm{F}} + \|(\mathbf{P}_0 - \mathbf{P})\mathbf{P}_0^{\mathrm{T}}\|_{\mathrm{F}} \leq (\|\mathbf{P}\|_{\mathrm{F}} + \|\mathbf{P}_0\|_{\mathrm{F}})\|\mathbf{P} - \mathbf{P}_0\|_{\mathrm{F}}\\ &\leq 2n\|\mathbf{P} - \mathbf{P}_0\|_{\mathrm{F}} \leq 2M\vartheta_n^{-1}\sqrt{n^3 d\log n}.\end{aligned}$$

Since $\lambda_d(\mathbf{P}_0\mathbf{P}_0^{\mathrm{T}}) = \sigma_d(\mathbf{X}_0\widetilde{\mathbf{X}}_0^{\mathrm{T}})^2 \geq n^2\lambda_d(\mathbf{\Delta}_0)\lambda_d(\widetilde{\mathbf{\Delta}}_0)/4$, it follows from Davis-Kahan theorem that there exists $\widetilde{\mathbf{W}} \in \mathbb{O}(d)$ such that

$$\|\mathbf{U} - \mathbf{U}_0\widetilde{\mathbf{W}}\|_{\mathrm{F}} \leq \sqrt{2d}\|\sin\Theta(\mathbf{U}, \mathbf{U}_0)\|_2 \leq \frac{2\sqrt{2d}}{\sigma_d(\mathbf{P}_0\mathbf{P}_0^{\mathrm{T}})}\|\mathbf{P}\mathbf{P}^{\mathrm{T}} - \mathbf{P}_0\mathbf{P}_0^{\mathrm{T}}\|_{\mathrm{F}} \lesssim \frac{1}{\vartheta_n}\sqrt{\frac{\log n}{n}}.$$

The proof is thus completed. $\square$

We are now in a position to prove Theorem 13.

Proof of Theorem 13. Assume that *a posteriori* the event $\{(1/n)\|\mathbf{X}\widetilde{\mathbf{X}}^{\mathrm{T}} - \mathbf{X}_0\widetilde{\mathbf{X}}_0^{\mathrm{T}}\|_{\mathrm{F}} \leq M\vartheta_n^{-1}\sqrt{n^{-1}d\log n}\}$ occurs. By Lemma 23 and the definition, there exists some constant $\widetilde{M} > 0$ such that

$$\begin{aligned}\|\mathbf{C}(\mathbf{U}) - \mathbf{U}_0\widetilde{\mathbf{W}}\|_{\mathrm{F}} &\leq \|\mathbf{C}(\mathbf{U}) - \mathbf{U}\|_{\mathrm{F}} + \|\mathbf{U} - \mathbf{U}_0\widetilde{\mathbf{W}}\|_{\mathrm{F}}\\ &\leq 2\|\mathbf{U} - \mathbf{U}_0\widetilde{\mathbf{W}}\|_{\mathrm{F}} \leq \frac{\xi\lambda_d^{1/4}(\mathbf{\Delta}_0)\lambda_d^{1/4}(\widetilde{\mathbf{\Delta}}_0)}{2\vartheta_n}\sqrt{\frac{\widetilde{M}\log n}{2n}}.\end{aligned}$$

Now we argue that the number of rows

$$\mathcal{V} = \left\{i \in [n] : \|\{\mathbf{C}(\mathbf{U})\}_{i*} - \widetilde{\mathbf{W}}^{\mathrm{T}}\mathbf{u}_{0i}\|_2 > \frac{\xi\lambda_d^{1/4}(\mathbf{\Delta}_0)\lambda_d^{1/4}(\widetilde{\mathbf{\Delta}}_0)}{2\sqrt{2n}}\right\}$$

is no greater than $\widetilde{M}\vartheta_n^{-2}\log n$ by contradiction. Assuming otherwise, then we obtain

$$\|\mathbf{C}(\mathbf{U}) - \mathbf{U}_0\widetilde{\mathbf{W}}\|_{\mathrm{F}}^2 > \left(\frac{\widetilde{M}\log n}{\vartheta_n^2}\right)\left\{\frac{\xi\lambda_d^{1/4}(\mathbf{\Delta}_0)\lambda_d^{1/4}(\widetilde{\mathbf{\Delta}}_0)}{2\sqrt{2n}}\right\}^2$$

$$= \left\{ \frac{\xi \lambda_d^{1/4}(\mathbf{\Delta}_0)\lambda_d^{1/4}}{2\vartheta_n} \sqrt{\frac{\widetilde{M}\log n}{2n}} \right\}^2,$$

contradicting with the previous observation. Namely, $|\mathcal{V}^c| \geq n - \widetilde{M}\vartheta_n^{-2}\log n$. Consequently, for any $i, j \in \mathcal{V}^c$ such that $\{\mathbf{C}(\mathbf{U})\}_{i*} = \{\mathbf{C}(\mathbf{U})\}_{j*}$, we see that

$$\begin{aligned} \|\mathbf{u}_{0i} - \mathbf{u}_{0j}\|_2 &= \|\widetilde{\mathbf{W}}^{\mathrm{T}}(\mathbf{u}_{0i} - \mathbf{u}_{0j})\|_2 \\ &\leq \|\{\mathbf{C}(\mathbf{U})\}_{i*} - \widetilde{\mathbf{W}}^{\mathrm{T}}\mathbf{u}_{0i}\|_2 + \|\{\mathbf{C}(\mathbf{U})\}_{j*} - \widetilde{\mathbf{W}}^{\mathrm{T}}\mathbf{u}_{0j}\|_2 \\ &\leq \frac{\xi \lambda_d^{1/4}(\mathbf{\Delta}_0)\lambda_d^{1/4}(\widetilde{\mathbf{\Delta}}_0)}{\sqrt{2n}}, \end{aligned}$$

implying that $\mathbf{x}_{0i} = \mathbf{x}_{0j}$ by Lemma 22. Now denote $\mathbf{u}_{01}^*, \ldots, \mathbf{u}_{0K}^*$ the K unique rows of $\mathbf{U}_0$. Note that $n_k \geq |\mathcal{V}|$ for all k, *i.e.*, $\{\mathbf{x}_{0i} : i \in \mathcal{V}^c\} = \{\mathbf{x}_{0k}^* : k \in [K]\}$ because $|\mathcal{V}|/n_k \asymp \vartheta_n^{-2}n^{-1}\log n \to 0$ by assumption, it follows that for each $k \in [K]$,

$$\mathcal{B}_k := B_{\|\cdot\|_2}\left\{ \widetilde{\mathbf{W}}\mathbf{u}_{0k}^*, \frac{\xi \lambda_d^{1/4}(\mathbf{\Delta}_0)\lambda_d^{1/4}(\widetilde{\mathbf{\Delta}}_0)}{2\sqrt{2n}} \right\}$$

contains at least one element of $\{\{\mathbf{C}(\mathbf{U})\}_{i*} : i \in \mathcal{V}^c\}$. Since $\mathcal{B}_k$'s are disjoint by assumption, and there are only K distinct rows in $\mathbf{C}(\mathbf{U})$, it follows directly from the pigeonhole principle that each $\mathcal{B}_k$ contains exactly one element of $[\{\mathbf{C}(\mathbf{U})\}_{i*} : i \in \mathcal{V}^c]$. Consequently, if $\mathbf{u}_{0i} = \mathbf{u}_{0j} = \mathbf{u}_{0k}^*$ for some $i, j \in \mathcal{V}^c$ and $k \in [K]$, then $\{\mathbf{C}(\mathbf{U})\}_{i*}, \{\mathbf{C}(\mathbf{U})\}_{j*} \in \mathcal{B}_k$, implying that $\{\mathbf{C}(\mathbf{X})\}_{i*} = \{\mathbf{C}(\mathbf{X})\}_{j*}$.

The above argument can be briefly stated as follows: $\mathbf{x}_{0i} = \mathbf{x}_{0j}$ if and only if $\{\mathbf{C}(\mathbf{X})\}_{i*} = \{\mathbf{C}(\mathbf{X})\}_{j*}$. This immediately implies that

$$\inf_{\sigma \in \mathcal{S}_K} d_H\{\sigma \circ \tau(\cdot; \mathbf{U}_0), \tau(\cdot; \mathbf{U})\} \leq \frac{\widetilde{M}\log n}{\vartheta_n^2},$$

and the proof is thus completed by invoking Theorem 12. $\square$

5.3 Proofs for Chapter 4

5.3.1 Proof of Theorem 15

Proof of Theorem 15. We begin the proof with writing down the likelihood function for $\mathbf{x}_i$:

$$\ell_{\mathbf{A}}(\mathbf{x}_i) = \sum_{j\neq i}^{n}\{A_{ij}\log(\mathbf{x}_i^{\mathrm{T}}\mathbf{x}_{0j}) + (1-A_{ij})\log(1-\mathbf{x}_i^{\mathrm{T}}\mathbf{x}_{0j})\}.$$

For convenience we denote the following functions:

$$\begin{aligned}
M_n(\mathbf{x}) &= \frac{1}{n}\ell_{\mathbf{A}}(\mathbf{x}) = \frac{1}{n}\sum_{j\neq i}^{n}\{A_{ij}\log(\mathbf{x}^{\mathrm{T}}\mathbf{x}_{0j}) + (1-A_{ij})\log(1-\mathbf{x}^{\mathrm{T}}\mathbf{x}_{0j})\},\\
M(\mathbf{x}) &= \mathbb{E}_0\{M_n(\mathbf{x})\} = \frac{1}{n}\sum_{j\neq i}^{n}\{\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j}\log(\mathbf{x}^{\mathrm{T}}\mathbf{x}_{0j}) + (1-\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})\log(1-\mathbf{x}^{\mathrm{T}}\mathbf{x}_{0j})\},\\
\mathbf{\Psi}_n(\mathbf{x}) &= \frac{\partial M_n}{\partial \mathbf{x}}(\mathbf{x}) = \frac{1}{n}\sum_{j\neq i}^{n}\left\{\frac{(A_{ij}-\mathbf{x}^{\mathrm{T}}\mathbf{x}_{0j})}{\mathbf{x}^{\mathrm{T}}\mathbf{x}_{0j}(1-\mathbf{x}^{\mathrm{T}}\mathbf{x}_{0j})}\right\}\mathbf{x}_{0j},\\
\mathbf{\Psi}(\mathbf{x}) &= \mathbb{E}_0\{\mathbf{\Psi}_n(\mathbf{x})\} = \mathbb{E}_0\left\{\frac{\partial M_n}{\partial \mathbf{x}}(\mathbf{x})\right\} = \frac{1}{n}\sum_{j\neq i}^{n}\left\{\frac{(\mathbf{x}_{0i}-\mathbf{x})^{\mathrm{T}}\mathbf{x}_{0j}}{\mathbf{x}^{\mathrm{T}}\mathbf{x}_{0j}(1-\mathbf{x}^{\mathrm{T}}\mathbf{x}_{0j})}\right\}\mathbf{x}_{0j}.
\end{aligned}$$

Denote Ψ_{nk} the kth component of $\mathbf{\Psi}_n$, *i.e.*,

$$\Psi_{nk}(\mathbf{x}) = \frac{1}{n}\sum_{j\neq i}^{n}\left\{\frac{(A_{ij}-\mathbf{x}^{\mathrm{T}}\mathbf{x}_{0j})}{\mathbf{x}^{\mathrm{T}}\mathbf{x}_{0j}(1-\mathbf{x}^{\mathrm{T}}\mathbf{x}_{0j})}\right\}x_{0jk},\quad k=1,2,\ldots,d,$$

where $\mathbf{x}_{0j} = [x_{0j1},\ldots,x_{0jd}]^{\mathrm{T}}\in\mathbb{R}^d$. Simple algebra shows that

$$\begin{aligned}
\frac{\partial^2 M_n}{\partial\mathbf{x}\partial\mathbf{x}^{\mathrm{T}}}(\mathbf{x}) &= -\frac{1}{n}\sum_{j\neq i}^{n}\frac{\mathbf{x}_{0j}\mathbf{x}_{0j}^{\mathrm{T}}}{\mathbf{x}^{\mathrm{T}}\mathbf{x}_{0j}(1-\mathbf{x}^{\mathrm{T}}\mathbf{x}_{0j})} - \frac{1}{n}\sum_{j\neq i}^{n}\frac{(A_{ij}-\mathbf{x}^{\mathrm{T}}\mathbf{x}_{0j})(1-2\mathbf{x}^{\mathrm{T}}\mathbf{x}_{0j})\mathbf{x}_{0j}\mathbf{x}_{0j}^{\mathrm{T}}}{\{\mathbf{x}^{\mathrm{T}}\mathbf{x}_{0j}(1-\mathbf{x}^{\mathrm{T}}\mathbf{x}_{0j})\}^2},\\
\frac{\partial^2 M}{\partial\mathbf{x}\partial\mathbf{x}^{\mathrm{T}}}(\mathbf{x}) &= -\frac{1}{n}\sum_{j\neq i}^{n}\left\{\frac{(\mathbf{x}^{\mathrm{T}}\mathbf{x}_{0j})^2 - 2\mathbf{x}^{\mathrm{T}}\mathbf{x}_{0i}\mathbf{x}^{\mathrm{T}}\mathbf{x}_{0j} + \mathbf{x}^{\mathrm{T}}\mathbf{x}_{0i}}{\mathbf{x}^{\mathrm{T}}\mathbf{x}_{0j}(1-\mathbf{x}^{\mathrm{T}}\mathbf{x}_{0j})}\right\}\mathbf{x}_{0j}\mathbf{x}_{0j}^{\mathrm{T}},\\
\frac{\partial^2 \Psi_{nk}}{\partial\mathbf{x}\partial\mathbf{x}^{\mathrm{T}}}(\mathbf{x}) &= \frac{1}{n}\sum_{j\neq i}^{n}\frac{x_{0jk}(1-2\mathbf{x}^{\mathrm{T}}\mathbf{x}_{0j})}{\{\mathbf{x}^{\mathrm{T}}\mathbf{x}_{0j}(1-\mathbf{x}^{\mathrm{T}}\mathbf{x}_{0j})\}^2}\mathbf{x}_{0j}\mathbf{x}_{0j}^{\mathrm{T}}\\
&\quad + \frac{1}{n}\sum_{j\neq i}^{n}\frac{x_{0jk}\{(1-2\mathbf{x}^{\mathrm{T}}\mathbf{x}_{0j}) + 2(A_{ij}-\mathbf{x}^{\mathrm{T}}\mathbf{x}_{0j})\}}{\{\mathbf{x}^{\mathrm{T}}\mathbf{x}_{0j}(1-\mathbf{x}^{\mathrm{T}}\mathbf{x}_{0j})\}^2}\mathbf{x}_{0j}\mathbf{x}_{0j}^{\mathrm{T}}\\
&\quad + \frac{1}{n}\sum_{j\neq i}^{n}\frac{x_{0jk}\{2(A_{ij}-\mathbf{x}^{\mathrm{T}}\mathbf{x}_{0j})(1-2\mathbf{x}^{\mathrm{T}}\mathbf{x}_{0j})^2\}}{\{\mathbf{x}^{\mathrm{T}}\mathbf{x}_{0j}(1-\mathbf{x}^{\mathrm{T}}\mathbf{x}_{0j})\}^3}\mathbf{x}_{0j}\mathbf{x}_{0j}^{\mathrm{T}}.
\end{aligned}$$

Clearly, Θ_n is compact and $M_n(\mathbf{x})$ is continuous. Therefore $\widehat{\mathbf{x}}_i = \arg\max_{\mathbf{x}\in\Theta_n} M_n(\mathbf{x})$ exists with probability one. Furthermore, by Shannon's lemma (see, for example, Lemma 2.2.1 in [68]), we know that $M(\mathbf{x})$ is maximized at $\mathbf{x} = \mathbf{x}_{0i}$. Since $\mathbf{x}^{\mathrm{T}}\mathbf{x}_{0j} \in [\delta, 1-\delta]$ for all $j \in [n]$, implying that

$$\begin{aligned}
-\frac{\partial^2 M}{\partial \mathbf{x}\partial \mathbf{x}^{\mathrm{T}}}(\mathbf{x}) &= \frac{1}{n}\sum_{j\neq i}^{n}\left\{\frac{(\mathbf{x}^{\mathrm{T}}\mathbf{x}_{0j})^2 - 2\mathbf{x}^{\mathrm{T}}\mathbf{x}_{0i}\mathbf{x}^{\mathrm{T}}\mathbf{x}_{0j} + \mathbf{x}^{\mathrm{T}}\mathbf{x}_{0i}}{\mathbf{x}^{\mathrm{T}}\mathbf{x}_{0j}(1-\mathbf{x}^{\mathrm{T}}\mathbf{x}_{0j})}\right\}\mathbf{x}_{0j}\mathbf{x}_{0j}^{\mathrm{T}} \\
&\succ \frac{1}{n}\sum_{j\neq i}^{n}\left\{\frac{(\mathbf{x}^{\mathrm{T}}\mathbf{x}_{0j})^2 - 2\mathbf{x}^{\mathrm{T}}\mathbf{x}_{0i}\mathbf{x}^{\mathrm{T}}\mathbf{x}_{0j} + (\mathbf{x}^{\mathrm{T}}\mathbf{x}_{0i})^2}{\mathbf{x}^{\mathrm{T}}\mathbf{x}_{0j}(1-\mathbf{x}^{\mathrm{T}}\mathbf{x}_{0j})}\right\}\mathbf{x}_{0j}\mathbf{x}_{0j}^{\mathrm{T}} \\
&= \frac{1}{n}\sum_{j\neq i}^{n}\left\{\frac{(\mathbf{x}^{\mathrm{T}}\mathbf{x}_{0j} - \mathbf{x}^{\mathrm{T}}\mathbf{x}_{0i})^2}{\mathbf{x}^{\mathrm{T}}\mathbf{x}_{0j}(1-\mathbf{x}^{\mathrm{T}}\mathbf{x}_{0j})}\right\}\mathbf{x}_{0j}\mathbf{x}_{0j}^{\mathrm{T}} \succeq \mathbf{O},
\end{aligned}$$

it follows that for all $\epsilon > 0$,

$$\sup_{\|\mathbf{x}-\mathbf{x}_{0i}\|>\epsilon} M(\mathbf{x}) < M(\mathbf{x}_{0i}), \tag{5.22}$$

since $\mathbf{x}_{0i}$ is in the interior of Θ_n and the Hessian of M is strictly negative definite for all $\mathbf{x} \in \Theta_n$.

We first claim that

$$\sup_{\mathbf{x}\in\Theta_n} |M_n(\mathbf{x}) - M(\mathbf{x})| \overset{\mathbb{P}_0}{\to} 0. \tag{5.23}$$

Define a stochastic process $\{J(\mathbf{x}) = M_n(\mathbf{x}) - M(\mathbf{x}) : \mathbf{x} \in \Theta_n\}$. Since for any $\mathbf{x}_1, \mathbf{x}_2 \in \Theta_n$, there exists a constant K_δ only depending on $\delta > 0$, such that

$$\begin{aligned}
\left|\log\left(\frac{\mathbf{x}_1^{\mathrm{T}}\mathbf{x}_{0j}}{1-\mathbf{x}_1^{\mathrm{T}}\mathbf{x}_{0j}}\right) - \log\left(\frac{\mathbf{x}_2^{\mathrm{T}}\mathbf{x}_{0j}}{1-\mathbf{x}_2^{\mathrm{T}}\mathbf{x}_{0j}}\right)\right| &\le \sup_{\mathbf{x}\in\Theta_n, j\in[n]}\left\|\frac{\partial}{\partial\mathbf{x}}\log\left(\frac{\mathbf{x}^{\mathrm{T}}\mathbf{x}_{0j}}{1-\mathbf{x}^{\mathrm{T}}\mathbf{x}_{0j}}\right)\right\|\|\mathbf{x}_1 - \mathbf{x}_2\| \\
&\le K_\delta\|\mathbf{x}_1 - \mathbf{x}_2\|,
\end{aligned}$$

it follows from Hoeffding's inequality that

$$\mathbb{P}_0\left(|J(\mathbf{x}_1) - J(\mathbf{x}_2)|\right) \le 2\exp\left(-\frac{2nt^2}{K_\delta^2\|\mathbf{x}_1-\mathbf{x}_2\|^2}\right),$$

implying that $J(\cdot)$ is a sub-Gaussian process with respect to $K_\delta n^{-1/2}\|\cdot\|$. Hence the packing entropy can also be bounded: There exists some large constant $C > 0$, such that

$$\log\mathcal{D}\left(\epsilon, \Theta_n, \frac{K_\delta}{\sqrt{n}}\|\cdot\|\right) \le d\log\left(\frac{C}{\epsilon\sqrt{n}}\right).$$

Hence, by the fact that $\sup_{\mathbf{x}_1,\mathbf{x}_2\in\Theta_n} K_\delta n^{-1/2}\|\mathbf{x}_1 - \mathbf{x}_2\| \leq cn^{-1/2}$ for some constant $c \in (0, C)$, a maximum inequality for sub-Gaussian process (see, for example, Corollary 8.5 in [91]), and the change of variable $u = \log\{C/(\epsilon\sqrt{n})\}$, we have

$$
\begin{aligned}
\mathbb{E}_0\left(\sup_{\mathbf{x}\in\Theta_n}|J(\mathbf{x})|\right) &\lesssim \mathbb{E}_0(|J(\mathbf{x}_{0i})|) + \int_0^{cn^{-1/2}}\sqrt{\log\mathcal{D}\left(\epsilon,\Theta_n,\frac{M}{\sqrt{n}}\|\cdot\|\right)}\mathrm{d}\epsilon\\
&\lesssim \sqrt{\mathrm{var}_0(J(\mathbf{x}_{0i}))} + \int_0^{cn^{-1/2}}\sqrt{\log\frac{C}{\epsilon\sqrt{n}}}\mathrm{d}\epsilon\\
&= \left\{\frac{1}{n^2}\sum_{j\neq i}^n \mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j}(1-\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})\log\frac{\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j}}{1-\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j}}\right\}^{1/2} + \frac{C}{\sqrt{n}}\int_{\log\frac{C}{c}}^{\infty}\sqrt{u}\mathrm{e}^{-u}\mathrm{d}u\\
&\to 0
\end{aligned}
$$

as $n\to\infty$. Therefore we conclude that $\sup_{\mathbf{x}\in\Theta_n}|J(\mathbf{x})| = o_{\mathbb{P}_0}(1)$.

In the proof below we shall drop the superscript (MLE) from $\widehat{\mathbf{x}}_i^{(\mathrm{MLE})}$ and write $\widehat{\mathbf{x}}_i = \widehat{\mathbf{x}}_i^{(\mathrm{MLE})}$ for short. We next use the claim (5.23) to show that $\widehat{\mathbf{x}}_i$ is consistent for $\mathbf{x}_{0i}$. The proof here is quite similar to that of Theorem 5.7 in [66] and presented here for completeness. In fact, this implies that $M_n(\mathbf{x}_{0i}) \overset{\mathbb{P}_0}{\to} M(\mathbf{x}_{0i})$. Furthermore, $\widehat{\mathbf{x}}_i$ is the maximizer of M_n, implying that

$$
\begin{aligned}
M(\mathbf{x}_{0i}) - M(\widehat{\mathbf{x}}_i) &= M_n(\mathbf{x}_{0i}) + o_{\mathbb{P}_0}(1) - M(\widehat{\mathbf{x}}_i) \leq M_n(\widehat{\mathbf{x}}_i) - M(\widehat{\mathbf{x}}_i) + o_{\mathbb{P}_0}(1)\\
&\leq \sup_{\mathbf{x}\in\Theta_n}|J(\mathbf{x})| + o_{\mathbb{P}_0}(1) = o_{\mathbb{P}_0}(1).
\end{aligned}
$$

This shows that $\mathbb{P}_0(M(\mathbf{x}_{0i}) - M(\widehat{\mathbf{x}}_i) \geq \eta) \to 0$ for all $\eta > 0$. Recall that by (5.22) for all $\epsilon > 0$, there exists some $\eta(\epsilon) > 0$, such that $\|\widehat{\mathbf{x}} - \mathbf{x}_{0i}\| > \epsilon$ implies $M(\widehat{\mathbf{x}}_i) \leq M(\mathbf{x}_{0i}) - \eta(\epsilon)$. Namely, for all $\epsilon > 0$, there exists some $\eta = \eta(\epsilon) > 0$ such that

$$
\mathbb{P}_0\left(\|\widehat{\mathbf{x}}_i - \mathbf{x}_{0i}\| > \epsilon\right) \leq \mathbb{P}_0\left(M(\mathbf{x}_{0i}) - M(\widehat{\mathbf{x}}_i) \geq \eta\right) \to 0
$$

as $n\to\infty$. This completes the proof of consistency of $\widehat{\mathbf{x}}_i$ for $\mathbf{x}_{0i}$.

We finally show the asymptotic normality of $\widehat{\mathbf{x}}_i$. Since $\widehat{\mathbf{x}}_i$ is consistent for $\mathbf{x}_{0i}$, it follows that with probability tending to one, $\widehat{\mathbf{x}}_i$ is in the interior of Θ_n since $\mathbf{x}_{0i}$ is. Assume this event occurs. By Taylor's expansion, we have, for $k = 1, 2, \ldots, d$, that

$$
0 = \Psi_{nk}(\widehat{\mathbf{x}}_i) = \Psi_{nk}(\mathbf{x}_{0i}) + \frac{\partial\Psi_{nk}}{\partial\mathbf{x}^{\mathrm{T}}}(\mathbf{x}_{0i})(\widehat{\mathbf{x}}_i - \mathbf{x}_{0i}) + \frac{1}{2}(\widehat{\mathbf{x}}_i - \mathbf{x}_{0i})^{\mathrm{T}}\left\{\frac{\partial^2\Psi_{nk}}{\partial\mathbf{x}\partial\mathbf{x}^{\mathrm{T}}}(\widetilde{\mathbf{x}})\right\}(\widehat{\mathbf{x}}_i - \mathbf{x}_{0i}),
$$

where $\widetilde{\mathbf{x}}$ lies on the line segment linking $\mathbf{x}_{0i}$ and $\widehat{\mathbf{x}}_{0i}$. Since for any $\mathbf{x} \in \Theta_n$,

$$\begin{aligned}\left\|\frac{\partial^2 \Psi_{nk}}{\partial \mathbf{x}\partial \mathbf{x}^{\mathrm{T}}}(\mathbf{x})\right\| &\leq \frac{1}{n}\sum_{j\neq i}^{n}\frac{\{1+2(1-\delta)\}\|\mathbf{x}_{0j}\|^2}{\delta^2(1-\delta)^2} + \frac{1}{n}\sum_{j\neq i}^{n}\frac{\{(3-2\delta)+2(2-\delta)\}\|\mathbf{x}_{0j}\|^2}{\delta^2(1-\delta)^2}\\ &\quad + \frac{1}{n}\sum_{j\neq i}^{n}\frac{2(2-\delta)(3-2\delta)^2\|\mathbf{x}_{0j}\|^2}{\delta^3(1-\delta)^3} \lesssim \frac{1}{n}\|\mathbf{X}_0\|_{\mathrm{F}}^2 \leq 1,\end{aligned}$$

it follows that the Hessian of $\Psi_{nk}(\widetilde{\mathbf{x}})$ is bounded in probability. Observe that,

$$\mathbb{E}_0\left\{\frac{\partial \mathbf{\Psi}_n(\mathbf{x}_{0i})}{\partial \mathbf{x}^{\mathrm{T}}}\right\} = -\frac{1}{n}\sum_{j\neq i}^{n}\frac{\mathbf{x}_{0j}\mathbf{x}_{0j}^{\mathrm{T}}}{\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j}(1-\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})},$$

and for any $s, t \in [d]$,

$$\mathrm{var}_0\left\{\frac{\partial \Psi_{ns}(\mathbf{x}_{0i})}{\partial x_t}\right\} = \frac{1}{n^2}\sum_{j\neq i}^{n}\frac{(1-2\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})^2(x_{0js}x_{0jt})^2}{\{\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j}(1-\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})\}^3} \to 0$$

as $n \to \infty$, it follows from the law of large numbers that

$$\frac{\partial \mathbf{\Psi}_n(\mathbf{x}_{0i})}{\partial \mathbf{x}^{\mathrm{T}}} = -\mathbf{G}_n(\mathbf{x}_{0i}) + o_{\mathbb{P}_0}(1).$$

Therefore, we conclude from the Taylor's expansion and $\widehat{\mathbf{x}}_i - \mathbf{x}_{0i} = o_{\mathbb{P}_0}(1)$ that

$$\begin{aligned}-\mathbf{\Psi}_n(\mathbf{x}_{0i}) &= \left\{-\mathbf{G}_n(\mathbf{x}_{0i}) + o_{\mathbb{P}_0}(1) + \frac{1}{2}(\widehat{\mathbf{x}}_i - \mathbf{x}_{0i})^{\mathrm{T}}O_{\mathbb{P}_0}(1)\right\}(\widehat{\mathbf{x}}_i - \mathbf{x}_{0i})\\ &= \{-\mathbf{G}_n(\mathbf{x}_{0i}) + o_{\mathbb{P}_0}(1)\}(\widehat{\mathbf{x}}_i - \mathbf{x}_{0i}).\end{aligned}$$

Namely,

$$\sqrt{n}(\widehat{\mathbf{x}}_i - \mathbf{x}_{0i}) = \{\mathbf{G}_n(\mathbf{x}_{0i}) + o_{\mathbb{P}_0}(1)\}^{-1}\left\{\frac{1}{\sqrt{n}}\sum_{j\neq i}^{n}\frac{(A_{ij} - \mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})\mathbf{x}_{0j}}{\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j}(1-\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})}\right\}.$$

Observe that

$$\begin{aligned}\sum_{j\neq i}^{n}\mathbb{E}_0\left\{\left\|\frac{1}{\sqrt{n}}\frac{(A_{ij} - \mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})\mathbf{x}_{0j}}{\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j}(1-\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})}\right\|^3\right\} &\leq \frac{1}{n^{3/2}}\sum_{j\neq i}^{n}\frac{(2-\delta)^3\|\mathbf{x}_{0j}\|^3}{\{\delta(1-\delta)\}^3} \to 0,\\ \mathrm{var}_0\left(\frac{1}{\sqrt{n}}\sum_{j\neq i}^{n}\frac{(A_{ij} - \mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})\mathbf{x}_{0j}}{\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j}(1-\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})}\right) &= \mathbf{G}_n(\mathbf{x}_{0i}) \to \mathbf{G}(\mathbf{x}_{0i}) \quad \text{as } n \to \infty,\end{aligned}$$

it follows from Lyapunov's central limit theorem that $\sqrt{n}(\widehat{\mathbf{x}}_i - \mathbf{x}_{0i}) \xrightarrow{\mathcal{L}} \mathrm{N}(\mathbf{0}, \mathbf{G}(\mathbf{x}_{0i})^{-1})$. The proof is thus completed. □

5.3.2 Proof of Theorem 16

Before proceeding to the proof of Theorem 16, we first present a collection of technical lemmas for bounding the remainder $\widehat{\mathbf{R}}_i$ in (4.9). The proofs of these lemmas are deferred to Section 5.3.5.

Lemma 24 *Let* $\mathbf{A} \sim \mathrm{RDPG}(\mathbf{X}_0)$ *and assume the conditions in Theorem 16 holds. Let an estimator* $\widetilde{\mathbf{X}} \in \mathbb{R}^{n\times d}$ *satisfy the approximate linearization property (4.8) with an orthogonal matrix* $\mathbf{W} \in \mathbb{O}(d)$*. Then*

$$\|\widetilde{\mathbf{X}}\mathbf{W} - \rho_n^{1/2}\mathbf{X}_0\|_{2\to\infty} = O_{\mathbb{P}_0}\left(\frac{(\log n)^{(1\vee\omega)/2}}{\sqrt{n\rho_n}}\right).$$

Lemma 25 *Let* $\mathbf{A} \sim \mathrm{RDPG}(\mathbf{X}_0)$ *with sparsity factor* ρ_n*, and assume the conditions of Theorem 16 hold. Let an estimator* $\widetilde{\mathbf{X}} \in \mathbb{R}^{n\times d}$ *satisfy the approximate linearization property (4.8) with an orthogonal matrix* $\mathbf{W} \in \mathbb{O}(d)$*. Then*

$$\max_{i\in[n]}\left\|\frac{1}{n\sqrt{\rho_n}}\sum_{j=1}^n \frac{\rho_n\mathbf{x}_{0j}\mathbf{x}_{0i}^{\mathrm{T}}}{\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j}(1-\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})}(\rho_n^{-1/2}\mathbf{W}^{\mathrm{T}}\widetilde{\mathbf{x}}_j - \mathbf{x}_{0j})\right\| = O_{\mathbb{P}_0}\left(\frac{(\log n)^{(1\vee\omega)/2}}{n\rho_n^{1/2}}\right).$$

Lemma 26 *Let* $\mathbf{A} \sim \mathrm{RDPG}(\mathbf{X}_0)$ *with sparsity factor* ρ_n *and assume the conditions of Theorem 16 hold.Let an estimator* $\widetilde{\mathbf{X}} \in \mathbb{R}^{n\times d}$ *satisfy the approximate linearization property (4.8) with an orthogonal matrix* $\mathbf{W} \in \mathbb{O}(d)$*. Suppose* $\{\boldsymbol{\alpha}_{ijk} : i,j\in[n], k\in[d]\}$ *is a collection of deterministic vectors in* $\mathbb{R}^d$ *with* $\sup_{i,j\in[n],k\in[d]}\|\boldsymbol{\alpha}_{ijk}\| < \infty$*. Then*

$$\max_{i\in[n]}\frac{1}{n\sqrt{\rho_n}}\left|\sum_{j=1}^n(A_{ij}-\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})\boldsymbol{\alpha}_{ijk}^{\mathrm{T}}(\rho_n^{-1/2}\mathbf{W}^{\mathrm{T}}\widetilde{\mathbf{x}}_i - \mathbf{x}_{0i})\right| = O_{\mathbb{P}_0}\left(\frac{(\log n)^{1/2+(1\vee\omega)/2}}{n\rho_n}\right).$$

Lemma 27 *Let* $\mathbf{A} \sim \mathrm{RDPG}(\mathbf{X}_0)$ *with sparsity factor* ρ_n*, and assume the conditions of Theorem 16 hold. Let an estimator* $\widetilde{\mathbf{X}} \in \mathbb{R}^{n\times d}$ *satisfy the approximate linearization property (4.8) with an orthogonal matrix* $\mathbf{W} \in \mathbb{O}(d)$*. Suppose* $\{\boldsymbol{\beta}_{ijk} : i,j\in[n], k\in[d]\}$ *is a collection of deterministic vectors in* $\mathbb{R}^d$ *such that* $\sup_{i,j\in[n],k\in[d]}\|\boldsymbol{\beta}_{ijk}\| < \infty$*. Then for each individual* $i\in[n]$*,*

$$\left|\frac{1}{n\sqrt{\rho_n}}\sum_{j=1}^n(A_{ij}-\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})(\rho_n^{-1/2}\mathbf{W}^{\mathrm{T}}\widetilde{\mathbf{x}}_j - \mathbf{x}_{0j})^{\mathrm{T}}\boldsymbol{\beta}_{ijk}\right| = O_{\mathbb{P}_0}\left(\frac{(\log n)^{\omega/2}}{n\rho_n^{3/2}}\right)$$

and

$$\sum_{i=1}^{n}\left\{\frac{1}{n\sqrt{\rho_n}}\sum_{j=1}^{n}(A_{ij}-\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})(\rho_n^{-1/2}\mathbf{W}^{\mathrm{T}}\widetilde{\mathbf{x}}_j-\mathbf{x}_{0j})^{\mathrm{T}}\boldsymbol{\beta}_{ijk}\right\}^2=O_{\mathbb{P}_0}\left(\frac{(\log n)^{\omega}}{n\rho_n^3}\right).$$

Proof of Theorem 16. Let $\mathbf{W}\in\mathbb{O}(d)$ satisfy (4.8). For any $\mathbf{X}=[\mathbf{x}_1,\ldots,\mathbf{x}_n]^{\mathrm{T}}\in\mathbb{R}^{n\times d}$, denote $\mathbf{H}_i(\mathbf{X})=(1/n)\sum_{j=1}^{n}\mathbf{x}_j\{(\mathbf{x}_i^{\mathrm{T}}\mathbf{x}_j)(1-\mathbf{x}_i^{\mathrm{T}}\mathbf{x}_j)\}^{-1}\mathbf{x}_j^{\mathrm{T}}$. By definition,

$$\begin{aligned}
&\mathbf{W}^{\mathrm{T}}\widehat{\mathbf{x}}_i-\rho_n^{1/2}\mathbf{x}_{0i}\\
&\quad=(\mathbf{W}^{\mathrm{T}}\widetilde{\mathbf{x}}_i-\rho_n^{1/2}\mathbf{x}_{0i})\\
&\qquad+\mathbf{W}^{\mathrm{T}}\mathbf{H}_i(\widetilde{\mathbf{X}})^{-1}\mathbf{W}\frac{1}{n\sqrt{\rho_n}}\sum_{j=1}^{n}\left\{\frac{(A_{ij}-\widetilde{\mathbf{x}}_i^{\mathrm{T}}\widetilde{\mathbf{x}}_j)(\rho_n^{-1/2}\mathbf{W}^{\mathrm{T}}\widetilde{\mathbf{x}}_j)}{\rho_n^{-1}\widetilde{\mathbf{x}}_i^{\mathrm{T}}\widetilde{\mathbf{x}}_j(1-\widetilde{\mathbf{x}}_i^{\mathrm{T}}\widetilde{\mathbf{x}}_j)}-\frac{(A_{ij}-\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})\mathbf{x}_{0j}}{\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j}(1-\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})}\right\}\\
&\qquad+\frac{1}{n\sqrt{\rho_n}}\sum_{j=1}^{n}\frac{(A_{ij}-\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})}{\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j}(1-\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})}(\mathbf{W}^{\mathrm{T}}\mathbf{H}_i(\widetilde{\mathbf{X}})^{-1}\mathbf{W})\mathbf{x}_{0j}\\
&\quad=(\mathbf{W}^{\mathrm{T}}\widetilde{\mathbf{x}}_i-\rho_n^{1/2}\mathbf{x}_{0i})\\
&\qquad+\mathbf{G}_n(\mathbf{x}_{0i})^{-1}\frac{1}{n\sqrt{\rho_n}}\sum_{j=1}^{n}\left\{\frac{(A_{ij}-\widetilde{\mathbf{x}}_i^{\mathrm{T}}\widetilde{\mathbf{x}}_j)(\rho_n^{-1/2}\mathbf{W}^{\mathrm{T}}\widetilde{\mathbf{x}}_j)}{\rho_n^{-1}\widetilde{\mathbf{x}}_i^{\mathrm{T}}\widetilde{\mathbf{x}}_j(1-\widetilde{\mathbf{x}}_i^{\mathrm{T}}\widetilde{\mathbf{x}}_j)}-\frac{(A_{ij}-\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})\mathbf{x}_{0j}}{\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j}(1-\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})}\right\}\\
&\qquad+\{\mathbf{W}^{\mathrm{T}}\mathbf{H}_i(\widetilde{\mathbf{X}})^{-1}\mathbf{W}-\mathbf{G}_n(\mathbf{x}_{0i})^{-1}\}\\
&\qquad\times\frac{1}{n\sqrt{\rho_n}}\sum_{j=1}^{n}\left\{\frac{(A_{ij}-\widetilde{\mathbf{x}}_i^{\mathrm{T}}\widetilde{\mathbf{x}}_j)(\rho_n^{-1/2}\mathbf{W}^{\mathrm{T}}\widetilde{\mathbf{x}}_j)}{\rho_n^{-1}\widetilde{\mathbf{x}}_i^{\mathrm{T}}\widetilde{\mathbf{x}}_j(1-\widetilde{\mathbf{x}}_i^{\mathrm{T}}\widetilde{\mathbf{x}}_j)}-\frac{(A_{ij}-\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})\mathbf{x}_{0j}}{\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j}(1-\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})}\right\}\\
&\qquad+\frac{1}{n\sqrt{\rho_n}}\sum_{j=1}^{n}\frac{(A_{ij}-\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})}{\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j}(1-\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})}\{\mathbf{W}^{\mathrm{T}}\mathbf{H}_i(\widetilde{\mathbf{X}})^{-1}\mathbf{W}-\mathbf{G}_n(\mathbf{x}_{0i})^{-1}\}\mathbf{x}_{0j}\\
&\qquad+\frac{1}{n\sqrt{\rho_n}}\sum_{j=1}^{n}\frac{(A_{ij}-\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})}{\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j}(1-\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})}\mathbf{G}_n(\mathbf{x}_{0i})^{-1}\mathbf{x}_{0j}\\
&=(\mathbf{W}^{\mathrm{T}}\widetilde{\mathbf{x}}_i-\rho_n^{1/2}\mathbf{x}_{0i})+\mathbf{G}_n(\mathbf{x}_{0i})^{-1}\mathbf{R}_{i1}+\mathbf{R}_{i2}\mathbf{R}_{i1}+\mathbf{R}_{i3}\\
&\quad+\frac{1}{n\sqrt{\rho_n}}\sum_{j=1}^{n}\frac{(A_{ij}-\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})}{\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j}(1-\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})}\mathbf{G}_n(\mathbf{x}_{0i})^{-1}\mathbf{x}_{0j},
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{R}_{i1}&=\frac{1}{n\sqrt{\rho_n}}\sum_{j=1}^{n}\left\{\frac{(A_{ij}-\widetilde{\mathbf{x}}_i^{\mathrm{T}}\widetilde{\mathbf{x}}_j)(\rho_n^{-1/2}\mathbf{W}^{\mathrm{T}}\widetilde{\mathbf{x}}_j)}{\rho_n^{-1}\widetilde{\mathbf{x}}_i^{\mathrm{T}}\widetilde{\mathbf{x}}_j(1-\rho_n\widetilde{\mathbf{x}}_i^{\mathrm{T}}\widetilde{\mathbf{x}}_j)}-\frac{(A_{ij}-\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})\mathbf{x}_{0j}}{\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j}(1-\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})}\right\},\\
\mathbf{R}_{i2}&=\mathbf{W}^{\mathrm{T}}\mathbf{H}_i(\widetilde{\mathbf{X}})^{-1}\mathbf{W}-\mathbf{G}_n(\mathbf{x}_{0i})^{-1},\\
\mathbf{R}_{i3}&=\frac{1}{n\sqrt{\rho_n}}\sum_{j=1}^{n}\frac{(A_{ij}-\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})}{\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j}(1-\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})}\mathbf{R}_{i2}\mathbf{x}_{0j}.
\end{aligned}$$

We first analyze $\mathbf{R}_{i1}$. Denote the function $\boldsymbol{\phi}_{ij}:\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}$ by

$$\boldsymbol{\phi}_{ij}(\mathbf{u},\mathbf{v})=\frac{(A_{ij}-\rho_n\mathbf{u}^{\mathrm{T}}\mathbf{v})\mathbf{v}}{\mathbf{u}^{\mathrm{T}}\mathbf{v}(1-\rho_n\mathbf{u}^{\mathrm{T}}\mathbf{v})},\quad i,j\in[n],$$

and let $\boldsymbol{\phi}_{ij} = [\phi_{ij1}, \ldots, \phi_{ijd}]^{\mathrm{T}}$. By Taylor's expansion, we have, If $\|\mathbf{u} - \mathbf{x}_{0i}\| < \epsilon$ and $\|\mathbf{v} - \mathbf{x}_{0j}\| < \epsilon$ for sufficiently small $\epsilon > 0$, and $\delta \leq \min_{i,j\in[n]} \mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j} \leq \max_{i,j\in[n]} \mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j} \leq 1 - \delta$ for some constant $\delta > 0$, then

$$
\begin{aligned}
&\phi_{ijk}(\mathbf{u}, \mathbf{v}) - \phi_{ijk}(\mathbf{x}_{0i}, \mathbf{x}_{0j}) \\
&\quad = -\left\{\frac{\rho_n x_{0jk}\mathbf{x}_{0j}}{\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j}(1 - \rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})}\right\}^{\mathrm{T}}(\mathbf{u} - \mathbf{x}_{0i}) - \left\{\frac{\rho_n \mathbf{x}_{0i} x_{0jk}}{\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j}(1 - \rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})}\right\}^{\mathrm{T}}(\mathbf{v} - \mathbf{x}_{0j}) \\
&\qquad - \left[\frac{x_{0jk}(A_{ij} - \rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})(1 - 2\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})\mathbf{x}_{0j}}{\{\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j}(1 - \rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})\}^2}\right]^{\mathrm{T}}(\mathbf{u} - \mathbf{x}_{0i}) \\
&\qquad + \left[\frac{(A_{ij} - \rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})\{\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j}(1 - \rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})\mathbf{e}_k - x_{0jk}(1 - 2\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})\mathbf{x}_{0i}\}}{\{\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j}(1 - \rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})\}^2}\right]^{\mathrm{T}}(\mathbf{v} - \mathbf{x}_{0j}) \\
&\qquad + R_{ijk},
\end{aligned}
$$

where $\max_{i,j\in[n],k\in[d]} |R_{ijk}| \leq C_\delta \max(\|\mathbf{u} - \mathbf{u}_0\|^2, \|\mathbf{v} - \mathbf{v}_0\|^2)$ for some constant δ only depending on δ. Applying the above fact to $\mathbf{R}_{i1}$, we derive

$$
\begin{aligned}
\mathbf{R}_{i1} &= \frac{1}{n\sqrt{\rho_n}}\sum_{j=1}^n \{\boldsymbol{\phi}_{ij}(\rho_n^{-1/2}\mathbf{W}^{\mathrm{T}}\widetilde{\mathbf{x}}_i, \rho_n^{-1/2}\mathbf{W}^{\mathrm{T}}\widetilde{\mathbf{x}}_j) - \boldsymbol{\phi}_{ij}(\mathbf{x}_{0i}, \mathbf{x}_{0j})\} \\
&= -\frac{1}{n\sqrt{\rho_n}}\sum_{j=1}^n \frac{\rho_n\mathbf{x}_{0j}\mathbf{x}_{0j}^{\mathrm{T}}}{\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j}(1 - \rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})}(\rho_n^{-1/2}\mathbf{W}^{\mathrm{T}}\widetilde{\mathbf{x}}_i - \mathbf{x}_{0i}) \\
&\quad - \frac{1}{n\sqrt{\rho_n}}\sum_{j=1}^n \frac{\rho_n\mathbf{x}_{0j}\mathbf{x}_{0i}^{\mathrm{T}}}{\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j}(1 - \rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})}(\rho_n^{-1/2}\mathbf{W}^{\mathrm{T}}\widetilde{\mathbf{x}}_j - \mathbf{x}_{0j}) \\
&\quad - \frac{1}{n\sqrt{\rho_n}}\sum_{j=1}^n \left[\frac{(A_{ij} - \rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})(1 - 2\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})\mathbf{x}_{0j}\mathbf{x}_{0j}^{\mathrm{T}}}{\{\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j}(1 - \rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})\}^2}\right](\rho_n^{-1/2}\mathbf{W}^{\mathrm{T}}\widetilde{\mathbf{x}}_i - \mathbf{x}_{0i}) \\
&\quad + \frac{1}{n\sqrt{\rho_n}}\sum_{j=1}^n \left[\frac{(A_{ij} - \rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{ij})\{\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j}(1 - \rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})\mathbf{I}_d - (1 - 2\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})\mathbf{x}_{0i}\mathbf{x}_{0j}^{\mathrm{T}}\}}{\{\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j}(1 - \rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})\}^2}\right]^{\mathrm{T}} \\
&\qquad\qquad \times (\rho_n^{-1/2}\mathbf{W}^{\mathrm{T}}\widetilde{\mathbf{x}}_j - \mathbf{x}_{0j}) \\
&\quad + \frac{1}{n\sqrt{\rho_n}}\sum_{j=1}^n \mathbf{R}_{ij} \\
&= -\mathbf{R}_{i11} - \mathbf{R}_{i12} - \mathbf{R}_{i13} + \mathbf{R}_{i14} + \frac{1}{n\sqrt{\rho_n}}\sum_{j=1}^n \mathbf{R}_{ij},
\end{aligned}
$$

where $\mathbf{R}_{ij}$'s are such that $\max_{i,j\in[n]} \|\mathbf{R}_{ij}\| \lesssim \|\rho_n^{-1/2}\widetilde{\mathbf{X}}\mathbf{W} - \mathbf{X}_0\|_{2\to\infty}^2$ when $\|\rho_n^{-1/2}\widetilde{\mathbf{X}}\mathbf{W} - \mathbf{X}_0\|_{2\to\infty}$ is sufficiently small. Clearly,

$$
\mathbf{R}_{i11} = \frac{1}{n}\sum_{j=1}^n \frac{\mathbf{x}_{0j}\mathbf{x}_{0j}^{\mathrm{T}}}{\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j}(1 - \rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})}(\mathbf{W}^{\mathrm{T}}\widetilde{\mathbf{x}}_i - \rho_n^{1/2}\mathbf{x}_{0i}) = \mathbf{G}_n(\mathbf{x}_{0i})(\mathbf{W}^{\mathrm{T}}\widetilde{\mathbf{x}}_i - \rho_n^{1/2}\mathbf{x}_{0i}).
$$

Furthermore, Lemma 25 shows that $\max_{i\in[n]}\|\mathbf{R}_{i12}\| = O_{\mathbb{P}_0}((n\sqrt{\rho_n})^{-1}(\log n)^{(1\vee\omega)/2})$. In addition, we have $\max_{i\in[n]}\|\mathbf{R}_{i13}\| = O_{\mathbb{P}_0}((n\rho_n)^{-1}(\log n)^{1/2+(1\vee\omega)/2})$ by Lemma 26, $\|\mathbf{R}_{i14}\| = O_{\mathbb{P}_0}((n\rho_n^{3/2})^{-1}(\log n)^{\omega/2})$ by Lemma 27, and

$$\max_{i\in[n]}\frac{1}{n\sqrt{\rho_n}}\sum_{j=1}^n\max_{j\in[n]}\|\mathbf{R}_{ij}\| = O_{\mathbb{P}_0}\left(\frac{(\log n)^{1\vee\omega}}{n\rho_n^{5/2}}\right)$$

by Lemma 24. This shows that

$$\begin{aligned}\|\mathbf{R}_{i1}+\mathbf{R}_{i11}\| &= \left\|\mathbf{R}_{i1}+\mathbf{G}_n(\mathbf{x}_{0i})(\mathbf{W}^{\mathrm{T}}\widetilde{\mathbf{x}}_i-\rho_n^{1/2}\mathbf{x}_{0i})\right\|\\ &\leq \|\mathbf{R}_{i12}\|+\|\mathbf{R}_{i13}\|+\|\mathbf{R}_{i14}\|+\frac{1}{n\sqrt{\rho_n}}\sum_{j=1}^n\max_{i,j\in[n]}\|\mathbf{R}_{ij}\| = O_{\mathbb{P}_0}\left(\frac{(\log n)^{1\vee\omega}}{n\rho_n^{5/2}}\right),\end{aligned}$$

and hence,

$$\|\mathbf{R}_{i1}\| \leq \|\mathbf{G}_n(\mathbf{x}_{0i})\|\|\widetilde{\mathbf{X}}\mathbf{W}-\rho_n^{1/2}\mathbf{X}_0\|_{2\to\infty}+O_{\mathbb{P}_0}\left(\frac{(\log n)^{1\vee\omega}}{n\rho_n^{5/2}}\right) = O_{\mathbb{P}_0}\left(\sqrt{\frac{(\log n)^{1\vee\omega}}{n\rho_n}}\right)$$

by Lemma 24.

Next we focus on $\mathbf{R}_{i2}$. Since the function $(\mathbf{u},\mathbf{v})\mapsto\{\mathbf{u}^{\mathrm{T}}\mathbf{v}(1-\rho_n\mathbf{u}^{\mathrm{T}}\mathbf{v})\}^{-1}\mathbf{v}\mathbf{v}^{\mathrm{T}}$ is Lipschitz continuous in a neighborhood of $(\mathbf{x}_{0i},\mathbf{x}_{0j})$, it follows immediately that

$$\max_{i\in[n]}\|\mathbf{W}^{\mathrm{T}}\mathbf{H}_i(\widetilde{\mathbf{X}})\mathbf{W}-\mathbf{G}_n(\mathbf{x}_{0i})\|_{\mathrm{F}} \lesssim \|\rho_n^{-1/2}\widetilde{\mathbf{X}}\mathbf{W}-\mathbf{X}_0\|_{2\to\infty}$$

when $\|\rho_n^{-1/2}\widetilde{\mathbf{X}}\mathbf{W}-\mathbf{X}_0\|_{2\to\infty}\leq C_c\rho_n^{-1}\sqrt{n^{-1}}(\log n)^{(1\vee\omega)/2}$. Namely,

$$\max_{i\in[n]}\|\mathbf{W}^{\mathrm{T}}\mathbf{H}_i(\widetilde{\mathbf{X}})\mathbf{W}-\mathbf{G}_n(\mathbf{x}_{0i})\|_{\mathrm{F}} = O_{\mathbb{P}_0}\left(\frac{(\log n)^{(1\vee\omega)/2}}{\rho_n\sqrt{n}}\right)$$

by Lemma 24. Furthermore, by the fact that $\mathbf{G}_n(\mathbf{x}_{0i})\to\mathbf{G}(\mathbf{x}_{0i})$ as $n\to\infty$, that $\mathbf{G}_n(\mathbf{x}_{0i})-\mathbf{\Delta}$ is positive definite for sufficiently large n, and that

$$|\lambda_d(\mathbf{W}^{\mathrm{T}}\mathbf{H}_i(\widetilde{\mathbf{X}})\mathbf{W})-\lambda_d(\mathbf{G}_n(\mathbf{x}_{0i}))| \leq \|\mathbf{W}^{\mathrm{T}}\mathbf{H}_i(\widetilde{\mathbf{X}})\mathbf{W}-\mathbf{G}_n(\mathbf{x}_{0i})\|_{\mathrm{F}}^2,$$

(see, for example, [92]), we conclude that

$$\min_{i\in[n]}\lambda_d(\mathbf{W}^{\mathrm{T}}\mathbf{H}_i(\widetilde{\mathbf{X}})\mathbf{W}) \geq \lambda_d(\mathbf{G}_n(\mathbf{x}_{0i}))-\max_{i\in[n]}\|\mathbf{W}^{\mathrm{T}}\mathbf{H}_i(\widetilde{\mathbf{X}})\mathbf{W}-\mathbf{G}_n(\mathbf{x}_{0i})\|_{\mathrm{F}}^2 \geq \lambda_d(\mathbf{\Delta})-o_{\mathbb{P}_0}(1),$$

namely, $\max_{i\in[n]}\lambda_d^{-1}(\mathbf{W}^{\mathrm{T}}\mathbf{H}_i(\widetilde{\mathbf{X}})\mathbf{W}) = O_{\mathbb{P}_0}(1)$. Therefore,

$$
\begin{aligned}
\max_{i\in[n]}\|\mathbf{R}_{i2}\|_{\mathrm{F}} &= \max_{i\in[n]}\|\{\mathbf{W}^{\mathrm{T}}\mathbf{H}_i(\widetilde{\mathbf{X}})\mathbf{W}\}^{-1}\{\mathbf{W}^{\mathrm{T}}\mathbf{H}_i(\widetilde{\mathbf{X}})\mathbf{W} - \mathbf{G}_n(\mathbf{x}_{0i})\}\mathbf{G}_n(\mathbf{x}_{0i})^{-1}\|_{\mathrm{F}}\\
&\lesssim \max_{i\in[n]}\lambda_d^{-1}(\mathbf{W}^{\mathrm{T}}\mathbf{H}_i(\widetilde{\mathbf{X}})\mathbf{W})\max_{i\in[n]}\|\mathbf{G}_n(\mathbf{x}_{0i})^{-1}\|_{\mathrm{F}}\max_{i\in[n]}\|\mathbf{W}^{\mathrm{T}}\mathbf{H}_i(\widetilde{\mathbf{X}})\mathbf{W} - \mathbf{G}_n(\mathbf{x}_{0i})\|_{\mathrm{F}}\\
&\leq O_{\mathbb{P}_0}(1)\|\boldsymbol{\Delta}^{-1}\|_{\mathrm{F}}\max_{i\in[n]}\|\mathbf{W}^{\mathrm{T}}\mathbf{H}_i(\widetilde{\mathbf{X}})\mathbf{W} - \mathbf{G}_n(\mathbf{x}_{0i})\|_{\mathrm{F}} = O_{\mathbb{P}_0}\left(\frac{(\log n)^{(1\vee\omega)/2}}{\rho_n\sqrt{n}}\right)
\end{aligned}
$$

We finally move forward to analyze $\mathbf{R}_{i3}$. Since

$$
\max_{i\in[n]}\|\mathbf{R}_{i3}\|_{\mathrm{F}} \leq \max_{i\in[n]}\|\mathbf{R}_{i2}\|_{\mathrm{F}}\sum_{k=1}^d\frac{1}{n\sqrt{\rho_n}}\max_{i\in[n]}\left|\sum_{j=1}^n\frac{(A_{ij}-\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})x_{0jk}}{\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j}(1-\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})}\right|,
$$

and by Hoeffding's inequality and the union bound,

$$
\begin{aligned}
&\mathbb{P}_0\left(\max_{i\in[d]}\left|\sum_{j=1}^n\frac{(A_{ij}-\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})x_{0jk}}{\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j}(1-\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})}\right| > t\sqrt{n\log n}\right)\\
&\quad = 2n\exp\left[-\frac{2t^2n\log n}{\sum_{j=1}^n x_{0jk}^2\{\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j}(1-\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})\}^{-2}}\right]\\
&\quad \leq 2\exp\left\{-(Mt^2-1)\log n\right\}
\end{aligned}
$$

for some constant $M > 0$. Hence,

$$
\sum_{k=1}^d\frac{1}{n\sqrt{\rho_n}}\max_{i\in[n]}\left|\sum_{j=1}^n\frac{(A_{ij}-\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})x_{0jk}}{\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j}(1-\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})}\right| = O_{\mathbb{P}_0}\left(\sqrt{\frac{\log n}{n\rho_n}}\right),
$$

and hence, $\max_{i\in[n]}\|\mathbf{R}_{i3}\|_{\mathrm{F}} = O_{\mathbb{P}_0}(\rho_n^{-3/2}n^{-1}(\log n)^{1/2+(1\vee\omega)/2})$. Therefore, we conclude that

$$
\begin{aligned}
\mathbf{W}^{\mathrm{T}}\widehat{\mathbf{x}}_i - \rho_n^{1/2}\mathbf{x}_{0i} &= (\mathbf{W}^{\mathrm{T}}\widetilde{\mathbf{x}}_i - \rho_n^{1/2}\mathbf{x}_{0i}) + \mathbf{G}_n(\mathbf{x}_{0i})^{-1}\mathbf{R}_1 + \mathbf{R}_2\mathbf{R}_1 + \mathbf{R}_3\\
&\quad + \frac{1}{n\sqrt{\rho_n}}\sum_{j=1}^n\frac{(A_{ij}-\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})}{\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j}(1-\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})}\mathbf{G}_n(\mathbf{x}_{0i})^{-1}\mathbf{x}_{0j}\\
&= \frac{1}{n\sqrt{\rho_n}}\sum_{j=1}^n\frac{(A_{ij}-\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})}{\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j}(1-\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})}\mathbf{G}_n(\mathbf{x}_{0i})^{-1}\mathbf{x}_{0j} + \widehat{\mathbf{R}}_i,
\end{aligned}
$$

where

$$
\begin{aligned}
\|\widehat{\mathbf{R}}_i\| &= \left\|\mathbf{G}_n(\mathbf{x}_{0i})^{-1}(\mathbf{R}_{i1} + \mathbf{R}_{i11}) + \mathbf{R}_{i2}\mathbf{R}_{i1} + \mathbf{R}_{i3}\right\|\\
&\leq \|\boldsymbol{\Delta}_n^{-1}\|_2\|\mathbf{R}_{i1} + \mathbf{R}_{i11}\| + (\|\mathbf{R}_{i2}\|_2\|\mathbf{R}_{i1}\| + \|\mathbf{R}_{i3}\|)
\end{aligned}
$$

$$
\begin{aligned}
&= O_{\mathbb{P}_0}\left(\frac{(\log n)^{1\vee\omega}}{n\rho_n^{5/2}}\right) + O_{\mathbb{P}_0}\left(\frac{(\log n)^{1\vee\omega}}{n\rho_n^{3/2}}\right) + O_{\mathbb{P}_0}\left(\frac{(\log n)^{1/2+(1\vee\omega)/2}}{n\rho_n^{3/2}}\right)\\
&= O_{\mathbb{P}_0}\left(\frac{(\log n)^{1\vee\omega}}{n\rho_n^{5/2}}\right).
\end{aligned}
$$

We now proceed to prove that $\sum_{i=1}^n \|\widehat{\mathbf{R}}_i\|^2 = O_{\mathbb{P}_0}((n\rho_n^5)^{-1}(\log n)^{2(1\vee\omega)})$. Observe that by Lemma 27 we have

$$
\begin{aligned}
&\sum_{i=1}^n \left\|\mathbf{G}_n(\mathbf{x}_{0i})^{-1}(-\mathbf{R}_{i12} - \mathbf{R}_{i13} + \mathbf{R}_{i14})\right\|^2\\
&\quad\le 3n\left\|\mathbf{G}_n(\mathbf{x}_{0i})^{-1}\right\|_{\mathrm{F}}^2\left\{\max_{i\in[n]}\|\mathbf{R}_{i12}\|^2 + \max_{i\in[n]}\|\mathbf{R}_{i13}\|^2\right\} + 3\sum_{i=1}^n\|\mathbf{R}_{i14}\|^2\\
&\quad\le n\left\|\mathbf{\Delta}^{-1}\right\|_{\mathrm{F}}^2\left\{\max_{i\in[n]}\|\mathbf{R}_{i12}\|^2 + \max_{i\in[n]}\|\mathbf{R}_{i13}\|^2\right\} + \sum_{i=1}^n\|\mathbf{R}_{i14}\|^2\\
&\quad= O_{\mathbb{P}_0}\left(\frac{(\log n)^{2(1\vee\omega)}}{n\rho_n^3}\right),
\end{aligned}
$$

and that

$$
\begin{aligned}
\sum_{i=1}^n\left\|\mathbf{G}_n(\mathbf{x}_{0i})^{-1}\frac{1}{n\sqrt{\rho_n}}\sum_{j=1}^n\mathbf{R}_{ij}\right\|^2 &\le n\|\mathbf{\Delta}^{-1}\|_{\mathrm{F}}^2\left(\frac{1}{n\sqrt{\rho_n}}\sum_{j=1}^n\max_{i\in[n]}\|\mathbf{R}_{ij}\|\right)^2\\
&= O_{\mathbb{P}_0}\left(\frac{(\log n)^{2(1\vee\omega)}}{n\rho_n^5}\right).
\end{aligned}
$$

Besides, by the above derivation we have

$$
\begin{aligned}
\sum_{i=1}^n\|\mathbf{R}_{i2}\mathbf{R}_{i1}\|^2 &\le \max_{i\in[n]}\|\mathbf{R}_{i2}\|_{\mathrm{F}}^2\sum_{i=1}^n\|\mathbf{R}_{i1}\|^2\\
&\le O_{\mathbb{P}_0}\left(\frac{(\log n)^{1\vee\omega}}{n^2\rho_n}\right)\left\{n\max_{i\in[n]}\|\mathbf{G}_n(\mathbf{x}_{0i})\|_{\mathrm{F}}^2\|\widetilde{\mathbf{X}}\mathbf{W} - \rho_n^{1/2}\mathbf{X}_0\|_{\mathrm{F}}^2\right\}\\
&\quad + O_{\mathbb{P}_0}\left(\frac{(\log n)^{1\vee\omega}}{n^2\rho_n}\right)\left\{\sum_{i=1}^n\left\|\mathbf{R}_{i12} + \mathbf{R}_{i13} - \mathbf{R}_{i14} - \frac{1}{n\sqrt{\rho_n}}\sum_{j=1}^n\mathbf{R}_{ij}\right\|^2\right\}\\
&= O_{\mathbb{P}_0}\left(\frac{(\log n)^{2(1\vee\omega)}}{n^2\rho_n^2}\right),
\end{aligned}
$$

and

$$
\sum_{i=1}^n\|\mathbf{R}_{i3}\|_{\mathrm{F}}^2 \le n\max_{i\in[n]}\|\mathbf{R}_{i3}\|_{\mathrm{F}}^2 = O_{\mathbb{P}_0}\left(\frac{(\log n)^{2(1\vee\omega)}}{n\rho_n^3}\right).
$$

Therefore, we conclude that

$$\begin{aligned}\sum_{i=1}^n \|\widehat{\mathbf{R}}_i\|_{\mathrm{F}}^2 &\leq 4\sum_{i=1}^n \left\|\mathbf{G}_n(\mathbf{x}_{0i})^{-1}(-\mathbf{R}_{i12} - \mathbf{R}_{i13} + \mathbf{R}_{i14})\right\|^2 \\ &\quad + 4\sum_{i=1}^n \left\|\mathbf{G}_n(\mathbf{x}_{0i})^{-1}\frac{1}{n\sqrt{\rho_n}}\sum_{j=1}^n \mathbf{R}_{ij}\right\|^2 \\ &\quad + 4\sum_{i=1}^n \|\mathbf{R}_{i2}\mathbf{R}_{i1}\|^2 + 4\sum_{i=1}^n \|\mathbf{R}_{i3}\|_{\mathrm{F}}^2 \\ &= O_{\mathbb{P}_0}\left(\frac{(\log n)^{2(1\vee\omega)}}{n\rho_n^5}\right).\end{aligned}$$

The proof is thus completed. □

5.3.3 Proof of Theorem 17

We begin the proof of Theorem 17 with the following two technical lemmas, the proofs of which are deferred to Section 5.3.5:

Lemma 28 *Let* $\mathbf{A} \sim \mathrm{RDPG}(\mathbf{X}_0)$ *and assume the conditions in Theorem 17 holds. Denote* $Z = Z(\mathbf{A}) = \sum_{i=1}^n \|\sum_{j=1}^n (A_{ij} - \rho_n \mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})\boldsymbol{\gamma}_{ij}\|^2$, *where* $\{\boldsymbol{\gamma}_{ij} : i, j \in [n]\}$ *is a collection of deterministic vectors in* $\mathbb{R}^d$ *such that* $\sup_{i,j\in[n]}\|\boldsymbol{\gamma}_{ij}\| \lesssim (n\sqrt{\rho_n})^{-1}$. *Then* $Z = \mathbb{E}_0(Z) + o_{\mathbb{P}_0}(1)$.

Lemma 29 *Let* $\mathbf{G}_n(\mathbf{x})$ *be defined as in Theorem 16,* $\mathbf{G}(\mathbf{x})$ *defined as in Theorem 15,* $\widetilde{\mathbf{G}}(\mathbf{x})$ *be defined as in Theorem 20. Denote*

$$\widetilde{\mathbf{G}}_n(\mathbf{x}) = \frac{1}{\boldsymbol{\mu}_n^{\mathrm{T}}\mathbf{x}}\left(\mathbf{I}_d - \frac{\mathbf{x}\boldsymbol{\mu}_n^{\mathrm{T}}}{2\mathbf{x}^{\mathrm{T}}\boldsymbol{\mu}_n}\right)\mathbf{G}_n(\mathbf{x})^{-1}\left(\mathbf{I}_d - \frac{\mathbf{x}\boldsymbol{\mu}_n^{\mathrm{T}}}{2\mathbf{x}^{\mathrm{T}}\boldsymbol{\mu}_n}\right),$$

where $\boldsymbol{\mu}_n = (1/n)\sum_{i=1}^n \mathbf{x}_{0i}$. *Let* $\mathcal{X}(\delta)$ *be the set of all* $\mathbf{x} \in \mathcal{X}$ *such that any* $\mathbf{x}, \mathbf{u} \in \mathcal{X}(\delta)$ *satisfy* $\delta \leq \mathbf{x}^{\mathrm{T}}\mathbf{u} \leq 1 - \delta$, *where* $\delta > 0$ *is some small constant independent of* n. *Then*

$$\sup_{\mathbf{x}\in\mathcal{X}(\delta)}\|\mathbf{G}_n(\mathbf{x})^{-1} - \mathbf{G}(\mathbf{x})^{-1}\|_{\mathrm{F}} \to 0, \quad \textit{and} \quad \sup_{\mathbf{x}\in\mathcal{X}(\delta)}\|\widetilde{\mathbf{G}}_n(\mathbf{x}) - \widetilde{\mathbf{G}}(\mathbf{x})\|_{\mathrm{F}} \to 0$$

as $n \to \infty$.

Proof of Theorem 17. Let $\mathbf{W}$ be the orthogonal matrix satisfying (4.8). Denote

$$\boldsymbol{\gamma}_{ij} = \frac{1}{n\sqrt{\rho_n}}\frac{\mathbf{G}_n(\mathbf{x}_{0i})^{-1}\mathbf{x}_{0j}}{\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j}(1 - \rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})}.$$

First note that $\mathbf{G}_n(\mathbf{x}_{0i})^{-1} \succeq \mathbf{\Delta}$ for sufficiently large n, and hence,

$$\sup_{i,j\in[n]} \|\boldsymbol{\gamma}_{ij}\| \leq \frac{1}{n\sqrt{\rho_n}} \sup_{i,j\in[n]} \frac{\|\mathbf{G}_n(\mathbf{x}_{0i})^{-1}\|\|\mathbf{x}_{0j}\|}{\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j}(1-\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})} \leq \frac{1}{n\sqrt{\rho_n}} \frac{\|\mathbf{\Delta}\|_2}{\delta(1-\delta)} \lesssim \frac{1}{n\sqrt{\rho_n}}. \tag{5.24}$$

Also observe that

$$\begin{aligned}
&\mathbb{E}_0\left(\sum_{i=1}^n\left\|\sum_{j=1}^n(A_{ij}-\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})\boldsymbol{\gamma}_{ij}\right\|^2\right)\\
&\quad=\sum_{i=1}^n\sum_{a=1}^n\sum_{b=1}^n\mathbb{E}_0\left\{(A_{ia}-\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0a})(A_{ib}-\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0b})\boldsymbol{\gamma}_{ia}^{\mathrm{T}}\boldsymbol{\gamma}_{ib}\right\}\\
&\quad=\frac{1}{n^2\rho_n}\sum_{i=1}^n\sum_{a=1}^n\frac{\mathbb{E}_0\{(A_{ia}-\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0a})^2\}}{\{\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0a}(1-\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0a})\}^2}\mathbf{x}_{0a}^{\mathrm{T}}\mathbf{G}_n^{-2}(\mathbf{x}_{0i})\mathbf{x}_{0a}\\
&\quad=\frac{1}{n}\sum_{i=1}^n\frac{1}{n}\sum_{a=1}^n\frac{\mathrm{tr}\{\mathbf{G}_n(\mathbf{x}_{0i})^{-1}\mathbf{x}_{0a}\mathbf{x}_{0a}^{\mathrm{T}}\mathbf{G}_n^{-1}(\mathbf{x}_{0i})\}}{\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0a}(1-\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0a})}\\
&\quad=\frac{1}{n}\sum_{i=1}^n\mathrm{tr}\left[\mathbf{G}_n(\mathbf{x}_{0i})^{-1}\left\{\frac{1}{n}\sum_{a=1}^n\frac{\mathbf{x}_{0a}\mathbf{x}_{0a}^{\mathrm{T}}}{\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0a}(1-\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0a})}\right\}\mathbf{G}_n(\mathbf{x}_{0i})^{-1}\right]\\
&\quad=\frac{1}{n}\sum_{i=1}^n\mathrm{tr}\{\mathbf{G}_n(\mathbf{x}_{0i})^{-1}\}.
\end{aligned}$$

By Theorem 16 and Lemma 28, we can write

$$\begin{aligned}
&\left\|\widehat{\mathbf{X}}\mathbf{W}-\mathbf{X}_0\right\|_{\mathrm{F}}^2\\
&\quad=\sum_{i=1}^n\left\|\sum_{j=1}^n(A_{ij}-\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})\boldsymbol{\gamma}_{ij}\right\|^2+2\sum_{i=1}^n\widehat{\mathbf{R}}_i^{\mathrm{T}}\sum_{j=1}^n(A_{ij}-\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})\boldsymbol{\gamma}_{ij}+\sum_{i=1}^n\|\widehat{\mathbf{R}}_i\|^2\\
&\quad=\frac{1}{n}\sum_{i=1}^n\mathrm{tr}\{\mathbf{G}_n(\mathbf{x}_{0i})^{-1}\}+2\sum_{i=1}^n\widehat{\mathbf{R}}_i^{\mathrm{T}}\sum_{j=1}^n(A_{ij}-\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})\boldsymbol{\gamma}_{ij}+o_{\mathbb{P}_0}(1).
\end{aligned}$$

By Cauchy-Schwarz inequality,

$$\begin{aligned}
\left|\sum_{i=1}^n\widehat{\mathbf{R}}_i^{\mathrm{T}}\sum_{j=1}^n(A_{ij}-\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})\boldsymbol{\gamma}_{ij}\right|&\leq\sum_{i=1}^n\|\widehat{\mathbf{R}}_i\|\left\|\sum_{j=1}^n(A_{ij}-\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})\gamma_{ijk}\right\|\\
&\leq\left(\sum_{i=1}^n\|\widehat{\mathbf{R}}_i\|^2\right)^{1/2}\left\{\sum_{i=1}^n\left\|\sum_{j=1}^n(A_{ij}-\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})\boldsymbol{\gamma}_{ij}\right\|^2\right\}^{1/2}\\
&=O_{\mathbb{P}_0}\left(\frac{(\log n)^{1\vee\omega}}{\sqrt{n\rho_n^5}}\right)\left\{\frac{1}{n}\sum_{i=1}^n\mathrm{tr}\{\mathbf{G}_n(\mathbf{x}_{0i})^{-1}\}+o_{\mathbb{P}_0}(1)\right\}^{1/2}\\
&=o_{\mathbb{P}_0}(1).
\end{aligned}$$

Hence, by condition (4.1) and the uniform convergence of $\mathbf{G}_n(\mathbf{x})^{-1} \to \mathbf{G}(\mathbf{x})^{-1}$ for all $\mathbf{x}$ (Lemma 29), we obtain (see, for example, Exercise 3 in Section 4.4 of [93])

$$\frac{1}{n}\sum_{i=1}^{n}\mathrm{tr}\{\mathbf{G}_n(\mathbf{x}_{0i})\} = \int \mathrm{tr}\{\mathbf{G}_n(\mathbf{x})^{-1}\}F_n(\mathrm{d}\mathbf{x}) \to \int_{\mathcal{X}} \mathrm{tr}\{\mathbf{G}(\mathbf{x})^{-1}\}F(\mathrm{d}\mathbf{x}).$$

This completes the first part of the theorem. For the second part, we observe that

$$\begin{aligned}
&\sum_{j=1}^{n}\mathbb{E}_0\left\{\left\|\frac{1}{\sqrt{n\rho_n}}\frac{(A_{ij}-\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})}{\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j}(1-\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})}\mathbf{G}_n(\mathbf{x}_{0i})^{-1}\mathbf{x}_{0j}\right\|^3\right\}\\
&\quad\le \frac{1}{(n\rho_n)^{3/2}}\sum_{j=1}^{n}\frac{\mathbb{E}_0\{(A_{ij}-\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})^3\}}{\{\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j}(1-\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})\}^3}\|\mathbf{G}_n^{-1}(\mathbf{x}_{0i})\|_2^3\|\mathbf{x}_{0j}\|^3\\
&\quad\lesssim \frac{1}{\sqrt{n\rho_n}} \lesssim \frac{(\log n)^{1\vee\omega}}{\sqrt{n\rho_n^5}} \to 0,\\
&\sum_{j=1}^{n}\mathrm{var}_0\left\{\frac{1}{\sqrt{n\rho_n}}\frac{(A_{ij}-\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})}{\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j}(1-\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})}\mathbf{G}_n(\mathbf{x}_{0i})^{-1}\mathbf{x}_{0j}\right\}\\
&\quad= \sum_{j\neq i}^{n}\mathrm{var}_0\left\{\frac{1}{\sqrt{n\rho_n}}\frac{(A_{ij}-\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})}{\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j}(1-\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})}\mathbf{G}_n(\mathbf{x}_{0i})^{-1}\mathbf{x}_{0j}\right\}\\
&\quad= \frac{1}{n\rho_n}\sum_{j\neq i}^{n}\frac{\rho_n}{\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j}(1-\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})}\mathbf{G}_n(\mathbf{x}_{0i})^{-1}\mathbf{x}_{0j}\mathbf{x}_{0j}^{\mathrm{T}}\mathbf{G}_n(\mathbf{x}_{0i})^{-1}\\
&\quad= \mathbf{G}_n(\mathbf{x}_{0i})^{-1} \to \mathbf{G}(\mathbf{x}_{0i})^{-1}.
\end{aligned}$$

It follows from the Lyapunov's central limit theorem and Theorem 16 that

$$\begin{aligned}
\sqrt{n}(\mathbf{W}^{\mathrm{T}}\widehat{\mathbf{x}}_i-\mathbf{x}_{0i}) &= \sum_{j=1}^{n}\frac{1}{\sqrt{n\rho_n}}\frac{(A_{ij}-\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})}{\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j}(1-\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})}\mathbf{G}_n(\mathbf{x}_{0i})^{-1}\mathbf{x}_{0j}+o_{\mathbb{P}_0}(1)\\
&\overset{\mathcal{L}}{\to} \mathrm{N}(\mathbf{0},\mathbf{G}(\mathbf{x}_{0i})^{-1}).
\end{aligned}$$

The proof is thus completed. □

5.3.4 Proof of Theorems 19 and 20

Proof of Theorem 19. Let $\mathbf{W}$ be the orthogonal matrix satisfying (4.8). Define a function $\mathbf{h}:\mathbb{R}^n\times\mathbb{R}^{n\times d}\to\mathbb{R}^d$ by

$$\mathbf{h}(\mathbf{x},\mathbf{T}) = [h_1(\mathbf{x},\mathbf{T}),\ldots,h_d(\mathbf{x},\mathbf{T})]^{\mathrm{T}} = \frac{\mathbf{x}}{\sqrt{(1/n)\sum_{j=1}^{n}\mathbf{x}^{\mathrm{T}}\mathbf{t}_j}},$$

where $\mathbf{T} = [\mathbf{t}_1, \ldots, \mathbf{t}_n]^{\mathrm{T}} \in \mathbb{R}^{n\times d}$. Simple algebra shows that for $k = 1, \ldots, d$

$$
\begin{aligned}
\frac{\partial h_k}{\partial \mathbf{x}^{\mathrm{T}}}(\rho_n^{1/2}\mathbf{x}_{0i}, \rho_n^{1/2}\mathbf{X}_0) &= \rho_n^{-1/2}\left(\mathbf{x}_{0i}^{\mathrm{T}}\boldsymbol{\mu}_n\right)^{-3/2}\left(\frac{1}{n}\sum_{j=1}^n \mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j}\mathbf{e}_k^{\mathrm{T}} - \frac{1}{2n}\sum_{j=1}^n \mathbf{e}_k^{\mathrm{T}}\mathbf{x}_{0i}\mathbf{x}_{0j}^{\mathrm{T}}\right),\\
\frac{\partial h_k}{\partial \mathbf{t}_j^{\mathrm{T}}}(\rho_n^{1/2}\mathbf{x}_{0i}, \rho_n^{1/2}\mathbf{X}_0) &= -\frac{1}{2n\sqrt{\rho_n}}\left(\mathbf{x}_{0i}^{\mathrm{T}}\boldsymbol{\mu}_n\right)^{-3/2}\mathbf{e}_k^{\mathrm{T}}\mathbf{x}_{0i}\mathbf{x}_{0i}^{\mathrm{T}},\\
\frac{\partial^2 h_k}{\partial \mathbf{x}\partial \mathbf{x}^{\mathrm{T}}}(\rho_n^{1/2}\mathbf{x}_{0i}, \rho_n^{1/2}\mathbf{X}_0) &= \rho_n^{-1}(\boldsymbol{\mu}_n^{\mathrm{T}}\mathbf{x}_{0i})^{-3/2}\\
&\quad\times\left\{-\frac{1}{2}\mathbf{e}_k\boldsymbol{\mu}_n^{\mathrm{T}} - \frac{1}{2}\boldsymbol{\mu}_n\mathbf{e}_k^{\mathrm{T}} + \frac{3}{4}(\mathbf{e}_k^{\mathrm{T}}\mathbf{x}_{0i})(\boldsymbol{\mu}_n^{\mathrm{T}}\mathbf{x}_{0i})^{-1}\boldsymbol{\mu}_n\boldsymbol{\mu}_n^{\mathrm{T}}\right\},\\
\frac{\partial^2 h_k}{\partial \mathbf{x}\partial \mathbf{t}_j^{\mathrm{T}}}(\rho_n^{1/2}\mathbf{x}_{0i}, \rho_n^{1/2}\mathbf{X}_0) &= -\frac{1}{2n\rho_n}(\boldsymbol{\mu}_n^{\mathrm{T}}\mathbf{x}_{0i})^{-3/2}\\
&\quad\times\left\{\mathbf{e}_k\mathbf{x}_{0i}^{\mathrm{T}} + (\mathbf{e}_k^{\mathrm{T}}\mathbf{x}_{0i})\mathbf{I} - \frac{3}{2n}(\mathbf{e}_k^{\mathrm{T}}\mathbf{x}_{0i})(\boldsymbol{\mu}_n^{\mathrm{T}}\mathbf{x}_{0i})^{-1}\mathbf{x}_{0i}\mathbf{x}_{0i}^{\mathrm{T}}\right\},\\
\frac{\partial^2 h_k}{\partial \mathbf{t}_l\partial \mathbf{t}_j^{\mathrm{T}}}(\rho_n^{1/2}\mathbf{x}_{0i}, \rho_n^{1/2}\mathbf{X}_0) &= \frac{3}{4n^2\rho_n}(\mathbf{e}_k^{\mathrm{T}}\mathbf{x}_{0i})(\boldsymbol{\mu}_n^{\mathrm{T}}\mathbf{x}_{0i})^{-5/2}\mathbf{x}_{0i}\mathbf{x}_{0i}^{\mathrm{T}}.
\end{aligned}
$$

Note that

$$
\begin{aligned}
\sup_{j\in[n]}\left\|\frac{\partial^2 h_k}{\partial \mathbf{x}\partial \mathbf{t}_j^{\mathrm{T}}}(\rho_n^{1/2}\mathbf{x}_{0i}, \rho_n^{1/2}\mathbf{X}_0)\right\|_{\mathrm{F}} &= O\left(\frac{1}{n\rho_n}\right),\\
\sup_{j,l\in[n]}\left\|\frac{\partial^2 h_k}{\partial \mathbf{t}_l\partial \mathbf{t}_j^{\mathrm{T}}}(\rho_n^{1/2}\mathbf{x}_{0i}, \rho_n^{1/2}\mathbf{X}_0)\right\|_{\mathrm{F}} &= O\left(\frac{1}{n^2\rho_n}\right).
\end{aligned}
$$

It follows from Taylor's expansion that

$$
\begin{aligned}
\mathbf{h}(\mathbf{W}^{\mathrm{T}}\widehat{\mathbf{x}}_i, \widetilde{\mathbf{X}}\mathbf{W}) &= \mathbf{h}(\rho_n^{1/2}\mathbf{x}_{0i}, \rho_n^{1/2}\mathbf{X}_0) + \frac{\partial \mathbf{h}}{\partial \mathbf{x}^{\mathrm{T}}}(\rho_n^{1/2}\mathbf{x}_{0i}, \rho_n^{1/2}\mathbf{X}_0)(\mathbf{W}^{\mathrm{T}}\widehat{\mathbf{x}}_i - \rho_n^{1/2}\mathbf{x}_{0i})\\
&\quad + \sum_{j=1}^n \frac{\partial \mathbf{h}}{\partial \mathbf{t}_j^{\mathrm{T}}}(\rho_n^{1/2}\mathbf{x}_{0i}, \rho_n^{1/2}\mathbf{X}_0)(\mathbf{W}^{\mathrm{T}}\widetilde{\mathbf{x}}_j - \rho_n^{1/2}\mathbf{x}_{0j})\\
&\quad + \mathbf{R}_{\mathbf{x}_i} + \sum_{j=1}^n \mathbf{R}_{\mathbf{x}_i\mathbf{t}_j} + \sum_{j=1}^n\sum_{l=1}^n \mathbf{R}_{\mathbf{t}_j\mathbf{t}_l},
\end{aligned}
$$

where

$$
\max_{i\in[n]}\|\mathbf{R}_{\mathbf{x}_i}\| \lesssim \frac{(\log n)^{1\vee\omega}}{\rho_n^3 n},\quad \sup_{i,j\in[n]}\|\mathbf{R}_{\mathbf{x}_i\mathbf{t}_j}\| \lesssim \frac{(\log n)^{1\vee\omega}}{n^2\rho_n^3},\quad \sup_{j,l\in[n]}\|\mathbf{R}_{\mathbf{t}_l\mathbf{t}_j}\| \lesssim \frac{(\log n)^{1\vee\omega}}{n^3\rho_n^3}
$$

when

$$
\|\widetilde{\mathbf{X}}\mathbf{W} - \rho_n^{1/2}\mathbf{X}_0\|_{2\to\infty} \leq C_c\frac{(\log n)^{(1\vee\omega)/2}}{\sqrt{n\rho_n}},\quad \|\widehat{\mathbf{X}}\mathbf{W} - \rho_n^{1/2}\mathbf{X}_0\|_{2\to\infty} \leq C_c\frac{(\log n)^{(1\vee\omega)/2}}{\sqrt{n\rho_n}}
$$

for some constant $C_c > 0$. Note that by Theorem 16, we have

$$\|\widehat{\mathbf{X}}\mathbf{W} - \rho_n^{1/2}\mathbf{X}_0\|_{2\to\infty} \leq \sum_{k=1}^d \max_{i\in[n]}\left|\frac{1}{n\sqrt{\rho_n}}\sum_{j=1}^n \frac{(A_{ij} - \rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})[\mathbf{G}_n(\mathbf{x}_{0i})^{-1}\mathbf{x}_{0j}]_k}{\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j}(1 - \rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})}\right| + O_{\mathbb{P}_0}\left(\frac{(\log n)^{1\vee\omega}}{n\rho_n^{5/2}}\right).$$

By Hoeffding's inequality and the union bound, we see that

$$\max_{i\in[n]}\left|\frac{1}{n\sqrt{\rho_n}}\sum_{j=1}^n \frac{(A_{ij} - \rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})[\mathbf{G}_n(\mathbf{x}_{0i})^{-1}\mathbf{x}_{0j}]_k}{\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j}(1 - \rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})}\right| = O_{\mathbb{P}_0}\left(\sqrt{\frac{\log n}{n\rho_n}}\right).$$

Thus, we conclude that $\|\widehat{\mathbf{X}}\mathbf{W} - \rho_n^{1/2}\mathbf{X}_0\|_{2\to\infty} = O_{\mathbb{P}_0}((n\rho_n)^{-1/2}(\log n)^{(1\vee\omega)/2})$. Invoking this fact and Lemma 24, we see that

$$\begin{aligned}
\sqrt{n}(\mathbf{W}^{\mathrm{T}}\widehat{\mathbf{y}}_i - \mathbf{y}_{0i}) &= \mathbf{h}(\mathbf{W}^{\mathrm{T}}\widehat{\mathbf{x}}_i, \widetilde{\mathbf{X}}\mathbf{W}) - \mathbf{h}(\rho_n^{1/2}\mathbf{x}_{0i}, \rho_n^{1/2}\mathbf{X}_0)\\
&= \rho_n^{-1/2}\left(\boldsymbol{\mu}_n^{\mathrm{T}}\mathbf{x}_{0i}\right)^{-3/2}\left\{\frac{1}{n}\sum_{j=1}^n\left(\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j}\mathbf{I}_d - \frac{1}{2}\mathbf{x}_{0i}\mathbf{x}_{0j}^{\mathrm{T}}\right)\right\}(\mathbf{W}^{\mathrm{T}}\widehat{\mathbf{x}}_i - \rho_n^{1/2}\mathbf{x}_{0i})\\
&\quad + \mathbf{R}_{i1}^{(\mathrm{L})} + \mathbf{R}_{i2}^{(\mathrm{L})},
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{R}_{i1}^{(\mathrm{L})} &= \sum_{j=1}^n \boldsymbol{\Xi}_{ij}(\mathbf{W}^{\mathrm{T}}\widetilde{\mathbf{x}}_j - \rho_n^{1/2}\mathbf{x}_{0j}),\\
\boldsymbol{\Xi}_{ij} &= [\boldsymbol{\xi}_{ij1}, \ldots, \boldsymbol{\xi}_{ijd}]^{\mathrm{T}} = -\frac{1}{2n\sqrt{\rho_n}}\left(\mathbf{x}_{0i}^{\mathrm{T}}\boldsymbol{\mu}_n\right)^{-3/2}\mathbf{x}_{0i}\mathbf{x}_{0i}^{\mathrm{T}},
\end{aligned}$$

and

$$\max_{i\in[n]}\|\mathbf{R}_{i2}^{(\mathrm{L})}\| \leq \max_{i\in[n]}\|\mathbf{R}_{\mathbf{x}_i}\| + n\max_{j\in[n]}\|\mathbf{R}_{\mathbf{x}_i\mathbf{t}_j}\| + n^2\max_{j,l}\|\mathbf{R}_{\mathbf{t}_j\mathbf{t}_l}\| = O_{\mathbb{P}_0}\left(\frac{(\log n)^{1\vee\omega}}{n\rho_n^3}\right).$$

By an argument that is similar to the proof of Lemma 26, we see that

$$\max_{i\in[n]}\|\mathbf{R}_{i1}^{(\mathrm{L})}\| \lesssim \sum_{k=1}^d \max_{i\in[n]}\left|\sum_{j=1}^n \boldsymbol{\xi}_{ijk}^{\mathrm{T}}(\mathbf{W}^{\mathrm{T}}\widetilde{\mathbf{x}}_j - \rho_n^{1/2}\mathbf{x}_{0j})\right| = O_{\mathbb{P}_0}\left(\frac{(\log n)^{(1\vee\omega)/2}}{n\rho_n}\right).$$

Hence we conclude that

$$\begin{aligned}
\sqrt{n}(\mathbf{W}^{\mathrm{T}}\widehat{\mathbf{y}}_i - \mathbf{y}_{0i}) &= \rho_n^{-1/2}\left(\boldsymbol{\mu}_n^{\mathrm{T}}\mathbf{x}_{0i}\right)^{-3/2}\\
&\quad\times\left\{\frac{1}{n}\sum_{j=1}^n\left(\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j}\mathbf{I}_d - \frac{1}{2}\mathbf{x}_{0i}\mathbf{x}_{0j}^{\mathrm{T}}\right)\right\}(\mathbf{W}^{\mathrm{T}}\widehat{\mathbf{x}}_i - \rho_n^{1/2}\mathbf{x}_{0i}) + \mathbf{R}_i^{(\mathrm{L})}
\end{aligned}$$

$$= \rho_n^{-1/2} \frac{1}{\sqrt{\boldsymbol{\mu}_n^{\mathrm{T}} \mathbf{x}_{0i}}} \left(\mathbf{I}_d - \frac{\mathbf{x}_{0i} \boldsymbol{\mu}_n^{\mathrm{T}}}{2 \boldsymbol{\mu}_n^{\mathrm{T}} \mathbf{x}_{0i}} \right) (\mathbf{W}^{\mathrm{T}} \widehat{\mathbf{x}}_i - \rho_n^{1/2} \mathbf{x}_{0i}) + \mathbf{R}_i^{(\mathrm{L})},$$

where $\max_{i \in [n]} \|\mathbf{R}_i^{(\mathrm{L})}\| = O_{\mathbb{P}_0}((n\rho_n^3)^{-1} (\log n)^{1 \vee \omega})$. This further implies that

$$\sum_{i=1}^n \|\mathbf{R}_i^{(\mathrm{L})}\|^2 = O_{\mathbb{P}_0}\left((n\rho_n^6)^{-1} (\log n)^{2(1 \vee \omega)} \right).$$

The proof is thus completed. $\square$

Proof of Theorem 20. Let $\mathbf{W}$ be the orthogonal matrix satisfying (4.8). Denote

$$\boldsymbol{\gamma}_{ij} = \frac{1}{n\sqrt{\rho_n}} (\boldsymbol{\mu}_n^{\mathrm{T}} \mathbf{x}_{0i})^{-1/2} \left(\mathbf{I}_d - \frac{\mathbf{x}_{0i} \boldsymbol{\mu}_n^{\mathrm{T}}}{\boldsymbol{\mu}_n^{\mathrm{T}} \mathbf{x}_{0i}} \right) \frac{\mathbf{G}_n(\mathbf{x}_{0i})^{-1} \mathbf{x}_{0j}}{\mathbf{x}_{0i}^{\mathrm{T}} \mathbf{x}_{0j} (1 - \rho_n \mathbf{x}_{0i}^{\mathrm{T}} \mathbf{x}_{0j})}.$$

First note that $\|\mathbf{G}_n(\mathbf{x}_{0i})^{-1}\|_2 \leq \|\boldsymbol{\Delta}^{-1}\|_2$ for sufficiently large n, and hence,

$$\sup_{i,j \in [n]} \|\boldsymbol{\gamma}_{ij}\| \leq \frac{1}{n\sqrt{\rho_n}} \delta^{-1/2} \left(1 + \frac{1}{\delta} \right) \sup_{i,j \in [n]} \frac{\|\mathbf{G}_n(\mathbf{x}_{0i})^{-1}\| \|\mathbf{x}_{0j}\|}{\mathbf{x}_{0i}^{\mathrm{T}} \mathbf{x}_{0j} (1 - \rho_n \mathbf{x}_{0i}^{\mathrm{T}} \mathbf{x}_{0j})} \lesssim \frac{1}{n\sqrt{\rho_n}}. \tag{5.25}$$

Also observe that

$$\begin{aligned}
&\mathbb{E}_0 \left(\sum_{i=1}^n \left\| \sum_{j=1}^n (A_{ij} - \rho_n \mathbf{x}_{0i}^{\mathrm{T}} \mathbf{x}_{0j}) \boldsymbol{\gamma}_{ij} \right\|^2 \right) \\
&\quad = \sum_{i=1}^n \sum_{a=1}^n \sum_{b=1}^n \mathbb{E}_0 \left\{ (A_{ia} - \rho_n \mathbf{x}_{0i}^{\mathrm{T}} \mathbf{x}_{0a})(A_{ib} - \rho_n \mathbf{x}_{0i}^{\mathrm{T}} \mathbf{x}_{0b}) \boldsymbol{\gamma}_{ia}^{\mathrm{T}} \boldsymbol{\gamma}_{ib} \right\} \\
&\quad = \frac{1}{n} \sum_{i=1}^n \mathrm{tr}\{\widetilde{\mathbf{G}}_n(\mathbf{x}_{0i})\}.
\end{aligned}$$

Denote

$$\widehat{\mathbf{R}}_i^{(\mathrm{L})} = (\boldsymbol{\mu}_n^{\mathrm{T}} \mathbf{x}_{0i})^{-1/2} \left(\mathbf{I}_d - \frac{\mathbf{x}_{0i} \boldsymbol{\mu}_n^{\mathrm{T}}}{2 \mathbf{x}_{0i}^{\mathrm{T}} \boldsymbol{\mu}_n} \right) \widehat{\mathbf{R}}_i + \rho_n^{1/2} \mathbf{R}_i^{(\mathrm{L})}.$$

Clearly, $\sum_{i=1}^n \|\widehat{\mathbf{R}}_i^{(\mathrm{L})}\|^2 = O_{\mathbb{P}_0}((n\rho_n^5)^{-1} (\log n)^2)$. By Theorem 19 and Lemma 28, we can write

$$\begin{aligned}
&n\rho_n \left\| \widehat{\mathbf{Y}} \mathbf{W} - \mathbf{Y}_0 \right\|_{\mathrm{F}}^2 \\
&\quad = \sum_{i=1}^n \left\| \sum_{j=1}^n (A_{ij} - \rho_n \mathbf{x}_{0i}^{\mathrm{T}} \mathbf{x}_{0j}) \boldsymbol{\gamma}_{ij} \right\|^2 + 2 \sum_{i=1}^n (\widehat{\mathbf{R}}_i^{(\mathrm{L})})^{\mathrm{T}} \sum_{j=1}^n (A_{ij} - \rho_n \mathbf{x}_{0i}^{\mathrm{T}} \mathbf{x}_{0j}) \boldsymbol{\gamma}_{ij} + \sum_{i=1}^n \|\widehat{\mathbf{R}}_i^{(\mathrm{L})}\|^2 \\
&\quad = \frac{1}{n} \sum_{i=1}^n \mathrm{tr}\{\widetilde{\mathbf{G}}_n(\mathbf{x}_{0i})\} + 2 \sum_{i=1}^n (\widehat{\mathbf{R}}_i^{(\mathrm{L})})^{\mathrm{T}} \sum_{j=1}^n (A_{ij} - \rho_n \mathbf{x}_{0i}^{\mathrm{T}} \mathbf{x}_{0j}) \boldsymbol{\gamma}_{ij} + o_{\mathbb{P}_0}(1)
\end{aligned}$$

$$
\begin{aligned}
&\quad + O_{\mathbb{P}_0}\left(\frac{(\log n)^{2(1\vee\omega)}}{n\rho_n^5}\right)\\
&= \frac{1}{n}\sum_{i=1}^n \mathrm{tr}\{\widetilde{\mathbf{G}}_n(\mathbf{x}_{0i})\} + 2\sum_{i=1}^n (\widehat{\mathbf{R}}_i^{(\mathrm{L})})^{\mathrm{T}}\sum_{j=1}^n (A_{ij} - \rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})\boldsymbol{\gamma}_{ij} + o_{\mathbb{P}_0}(1).
\end{aligned}
$$

By Cauchy-Schwarz inequality and Lemma 28,

$$
\begin{aligned}
\left|\sum_{i=1}^n (\widehat{\mathbf{R}}_i^{(\mathrm{L})})^{\mathrm{T}}\sum_{j=1}^n (A_{ij} - \rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})\boldsymbol{\gamma}_{ij}\right| &\leq \sum_{i=1}^n \|\widehat{\mathbf{R}}_i^{(\mathrm{L})}\|\left\|\sum_{j=1}^n (A_{ij} - \rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})\gamma_{ijk}\right\|\\
&\leq \left(\sum_{i=1}^n \|\widehat{\mathbf{R}}_i^{(\mathrm{L})}\|^2\right)^{1/2}\left\{\sum_{i=1}^n\left\|\sum_{j=1}^n (A_{ij} - \rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})\boldsymbol{\gamma}_{ij}\right\|^2\right\}^{1/2}\\
&= o_{\mathbb{P}_0}(1).
\end{aligned}
$$

Furthermore, by condition (4.1) and Lemma 29, we see that

$$
\frac{1}{n}\sum_{i=1}^n \mathrm{tr}\{\widetilde{\mathbf{G}}_n(\mathbf{x}_{0i})\} = \int \mathrm{tr}\{\widetilde{\mathbf{G}}_n(\mathbf{x})\}F_n(\mathrm{d}\mathbf{x}) \to \int \mathrm{tr}\{\widetilde{\mathbf{G}}(\mathbf{x})\}F(\mathrm{d}\mathbf{x}).
$$

This completes the proof of the first part of the theorem. For the second part, we see that

$$
\frac{1}{\boldsymbol{\mu}_n^{\mathrm{T}}\mathbf{x}_{0i}}\left(\mathbf{I}_d - \frac{\mathbf{x}_{0i}\boldsymbol{\mu}_n^{\mathrm{T}}}{2\boldsymbol{\mu}_n^{\mathrm{T}}\mathbf{x}_{0i}}\right)\mathbf{G}(\mathbf{x}_{0i})^{-1}\left(\mathbf{I}_d - \frac{\mathbf{x}_{0i}\boldsymbol{\mu}_n^{\mathrm{T}}}{2\boldsymbol{\mu}_n^{\mathrm{T}}\mathbf{x}_{0i}}\right) = \widetilde{\mathbf{G}}_n(\mathbf{x}_{0i}) \to \widetilde{\mathbf{G}}(\mathbf{x}_{0i}).
$$

The result directly follows from the asymptotic normality of $\sqrt{n}(\mathbf{W}^{\mathrm{T}}\widehat{\mathbf{x}}_i - \rho_n^{1/2}\mathbf{x}_{0i})$. The proof is thus completed. □

5.3.5 Proofs of technical lemmas

Proof of Lemma 24. The proof of this lemma is similar to that of Lemma 2 in [39], except that we consider the case where a sparsity factor ρ_n is taken into account, and the proof is presented here for the sake of completeness. Recall from (4.8) that

$$
\|\widetilde{\mathbf{X}}\mathbf{W} - \rho_n^{1/2}\mathbf{X}_0\|_{2\to\infty} \leq \rho_n^{-1/2}\sqrt{d}\max_{i\in[n],k\in[d]}\left|\sum_{j=1}^n (A_{ij} - \rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})\zeta_{ijk}\right| + \|\widetilde{\mathbf{R}}\|_{\mathrm{F}}.
$$

where $\boldsymbol{\zeta}_{ij} = [\zeta_{ij1},\ldots,\zeta_{ijd}]^{\mathrm{T}} \in \mathbb{R}^d$. By Hoeffding's inequality, the union bound, and the condition that $\sup_{i,j\in[n]}\|\boldsymbol{\zeta}_{ij}\| \lesssim 1/n$, for any $t > 0$, we have,

$$
\mathbb{P}_0\left(\max_{i\in[n],k\in[k]}\left|\sum_{j=1}^n (A_{ij} - \rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})\zeta_{ijk}\right| > t\right) \leq 2nd\exp\left(-\frac{2t^2}{\sum_{j=1}^n \zeta_{ijk}^2}\right)
$$

$$= 2nd \exp\left(-Knt^2\right)$$

for some constant $K > 0$. Therefore, for any $c > 0$, there exists some constant $C_c > 0$ and $n_c \in \mathbb{N}_+$, such that for all $n \geq n_c$.

$$\mathbb{P}_0\left(\rho_n^{-1/2}\sqrt{d}\max_{i\in[n],k\in[d]}\left|\sum_{j=1}^n (A_{ij} - \rho_n \mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})\zeta_{ijk}\right| > C_c\sqrt{\frac{\log n}{n\rho_n}}\right) \leq \frac{1}{n^c}.$$

This shows that

$$\rho_n^{-1/2}\sqrt{d}\max_{i\in[n],k\in[d]}\left|\sum_{j=1}^n (A_{ij} - \rho_n \mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})\zeta_{ijk}\right| = O_{\mathbb{P}_0}\left(\sqrt{\frac{\log n}{n\rho_n}}\right).$$

The proof is completed by applying the condition that $\|\widetilde{\mathbf{R}}\|_{\mathrm{F}} = O_{\mathbb{P}_0}\left((n\rho_n)^{-1/2}(\log n)^{\omega/2}\right)$.

$\square$

Proof of Lemma 25. Recall by condition (4.8) that for any $j \in [n]$,

$$[\mathbf{W}^{\mathrm{T}}\widetilde{\mathbf{x}}_j - \rho_n^{1/2}\mathbf{x}_{0j}]_k = \rho_n^{-1/2}\sum_{a=1}^n (A_{ja} - \rho_n \mathbf{x}_{0j}^{\mathrm{T}}\mathbf{x}_{0a})\zeta_{iak} + \widetilde{R}_{jk}, \quad k = 1, 2, \ldots, d,$$

where $\boldsymbol{\zeta}_{ij} = [\zeta_{ij1}, \ldots, \zeta_{ijd}]^{\mathrm{T}}$. It follows that for $k = 1, 2, \ldots, d$,

$$\begin{aligned}
&\frac{1}{n\sqrt{\rho_n}}\sum_{j=1}^n\sum_{s=1}^d \frac{\rho_n x_{0jk}x_{0is}}{\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j}(1 - \rho_n \mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})}[\rho_n^{-1/2}\mathbf{W}^{\mathrm{T}}\widetilde{\mathbf{x}}_j - \mathbf{x}_{0j}]_s \\
&= \frac{1}{n\sqrt{\rho_n}}\sum_{s=1}^d\sum_{j=1}^n\sum_{a=1}^n \frac{x_{0jk}x_{0is}}{\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j}(1 - \rho_n \mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})}(A_{ja} - \rho_n \mathbf{x}_{0j}^{\mathrm{T}}\mathbf{x}_{0a})\zeta_{ias} \\
&\quad + \frac{1}{n}\sum_{j=1}^n\sum_{s=1}^d \frac{x_{0jk}x_{0is}}{\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j}(1 - \rho_n \mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})}\widetilde{R}_{js} \\
&= \frac{1}{n\sqrt{\rho_n}}\sum_{s=1}^d\left\{\sum_{j<a} Z_{iksja} + \sum_{j>a} Z_{iksja} + \sum_{j=1}^n Z_{iksjj}\right\} \\
&\quad + \frac{1}{n}\sum_{j=1}^n\sum_{s=1}^d \frac{x_{0jk}x_{0is}}{\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j}(1 - \rho_n \mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})}\widetilde{R}_{js},
\end{aligned}$$

where

$$Z_{iksja} = \frac{x_{0jk}x_{0is}}{\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j}(1 - \rho_n \mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})}(A_{ja} - \rho_n \mathbf{x}_{0j}^{\mathrm{T}}\mathbf{x}_{0a})\zeta_{ias}.$$

Observe that by Hoeffding's inequality and the union bound,

$$\mathbb{P}_0\left(\frac{1}{n\sqrt{\rho_n}}\max_{i\in[n]}\left|\sum_{j<a} Z_{iksja}\right| > t\right)$$

$$
\begin{aligned}
&\leq 2n\exp\left[-2n^2\rho_n t^2\left\{\sum_{j<a}\left(\frac{\zeta_{ias}x_{0jk}x_{0is}}{\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j}(1-\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})}\right)^2\right\}^{-1}\right]\\
&\leq 2n\exp\left(-Kn^2\rho_n t^2\right).
\end{aligned}
$$

This shows that

$$
\frac{1}{n\sqrt{\rho_n}}\max_{i\in[n]}\left|\sum_{j<a}Z_{iksja}\right| = O_{\mathbb{P}_0}\left(\sqrt{\frac{\log n}{n^2\rho_n}}\right),
$$

and hence, a similar argument yields that

$$
\frac{1}{n\sqrt{\rho_n}}\max_{i\in[n]}\left|\sum_{j=1}^n\sum_{a=1}^n Z_{iksja}\right| = O_{\mathbb{P}_0}\left(\sqrt{\frac{\log n}{n^2\rho_n}}\right).
$$

In addition, by the fact that $\|\widetilde{\mathbf{R}}\|_{\mathrm{F}}^2 = O_{\mathbb{P}_0}((n\rho_n)^{-1}(\log n)^{\omega})$ we have

$$
\begin{aligned}
&\max_{i\in[n]}\left|\frac{1}{n}\sum_{j=1}^n\sum_{s=1}^d\frac{x_{0jk}x_{0is}}{\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j}(1-\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})}\widetilde{R}_{js}\right|\\
&\quad\leq\frac{1}{n}\left[\sum_{j=1}^n\sum_{s=1}^d\max_{i\in[n]}\left\{\frac{x_{0jk}x_{0is}}{\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j}(1-\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})}\right\}^2\right]^{1/2}\|\widetilde{\mathbf{R}}\|_{\mathrm{F}}\\
&\quad\lesssim\frac{1}{\sqrt{n}}\|\widetilde{\mathbf{R}}\|_{\mathrm{F}} = O_{\mathbb{P}_0}\left(\frac{(\log n)^{\omega/2}}{n\rho_n^{1/2}}\right).
\end{aligned}
$$

Therefore, we conclude that

$$
\begin{aligned}
&\max_{i\in[n]}\left\|\frac{1}{n\sqrt{\rho_n}}\sum_{j=1}^n\frac{\rho_n\mathbf{x}_{0j}\mathbf{x}_{0i}^{\mathrm{T}}}{\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j}(1-\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})}(\rho_n^{-1/2}\mathbf{W}^{\mathrm{T}}\widetilde{\mathbf{x}}_j-\mathbf{x}_{0j})\right\|\\
&\lesssim\sum_{k=1}^d\sum_{s=1}^d\max_{i\in[n]}\left|\frac{1}{n\sqrt{\rho_n}}\sum_{j=1}^n\sum_{a=1}^n Z_{iksja}\right| + \sum_{k=1}^d\max_{i\in[n]}\left|\frac{1}{n}\sum_{j=1}^n\sum_{s=1}^d\frac{x_{0jk}x_{0is}}{\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j}(1-\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})}\widetilde{R}_{js}\right|\\
&= O_{\mathbb{P}_0}\left(\frac{(\log n)^{(1\vee\omega)/2}}{n\rho_n^{1/2}}\right),
\end{aligned}
$$

and the proof is thus completed. $\square$

Proof of Lemma 26. First observe that by Cauchy-Schwarz inequality,

$$
\begin{aligned}
&\left|\frac{1}{n\sqrt{\rho_n}}\sum_{j=1}^n(A_{ij}-\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})\boldsymbol{\alpha}_{ijk}^{\mathrm{T}}(\rho_n^{-1/2}\mathbf{W}^{\mathrm{T}}\widetilde{\mathbf{x}}_i-\mathbf{x}_{0i})\right|\\
&\quad\leq\frac{1}{\sqrt{\rho_n}}\|\mathbf{W}^{\mathrm{T}}\widetilde{\mathbf{x}}_i-\rho_n^{1/2}\mathbf{x}_{0i}\|\left\|\frac{1}{n\sqrt{\rho_n}}\sum_{j=1}^n\boldsymbol{\alpha}_{ijk}(A_{ij}-\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})\right\|.
\end{aligned}
$$

By Hoeffding's inequality and the union bound, for all $r = 1, 2, \ldots, d$,

$$\mathbb{P}_0\left(\max_{i\in[n]}\left|\frac{1}{n\sqrt{\rho_n}}\sum_{j=1}^n \alpha_{ijkr}(A_{ij} - \rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})\right| > t\right) \leq 2nd\exp\left(-Kn\rho_n t^2\right)$$

for some constant $K > 0$. This shows that

$$\max_{i\in[n]}\left|\frac{1}{n\sqrt{\rho_n}}\sum_{j=1}^n \alpha_{ijkr}(A_{ij} - \rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})\right| = O_{\mathbb{P}_0}\left(\sqrt{\frac{\log n}{n\rho_n}}\right)$$

Hence, we conclude from Lemma 24 that

$$\begin{aligned}
&\max_{i\in[n]}\left|\frac{1}{n\sqrt{\rho_n}}\sum_{j=1}^n (A_{ij} - \rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})\boldsymbol{\alpha}_{ijk}^{\mathrm{T}}(\rho_n^{-1/2}\mathbf{W}^{\mathrm{T}}\widetilde{\mathbf{x}}_i - \mathbf{x}_{0i})\right| \\
&\quad\lesssim \frac{1}{\sqrt{\rho_n}}\|\widetilde{\mathbf{X}}\mathbf{W} - \rho_n^{1/2}\mathbf{X}_0\|_{2\to\infty}\sum_{r=1}^d\left|\max_{i\in[n]}\frac{1}{n\sqrt{\rho_n}}\sum_{j=1}^n \alpha_{ijkr}(A_{ij} - \rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})\right| \\
&\quad= O_{\mathbb{P}_0}\left(\frac{(\log n)^{(1/2)+(1\vee\omega)/2}}{n\rho_n}\right).
\end{aligned}$$

The proof is thus completed. $\square$

Proof of Lemma 27. Denote $\boldsymbol{\beta}_{ijk} = [\beta_{ijk1}, \ldots, \beta_{ijkd}]^{\mathrm{T}}$. Recall the approximate linearization property (4.8) that

$$[\mathbf{W}^{\mathrm{T}}\widetilde{\mathbf{x}}_j - \rho_n^{1/2}\mathbf{x}_{0j}]_s = \rho_n^{-1/2}\sum_{a=1}^n (A_{ja} - \rho_n\mathbf{x}_{0j}^{\mathrm{T}}\mathbf{x}_{0a})\zeta_{ias} + \widetilde{R}_{js}, \quad s = 1, 2, \ldots, d,$$

where $\boldsymbol{\zeta}_{ia} = [\zeta_{ia1}, \ldots, \zeta_{iad}]^{\mathrm{T}}$. It follows that

$$\begin{aligned}
Q_{ik} &:= \frac{1}{n\sqrt{\rho_n}}\sum_{j=1}^n (A_{ij} - \rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})(\rho_n^{-1/2}\mathbf{W}^{\mathrm{T}}\widetilde{\mathbf{x}}_j - \mathbf{x}_{0j})^{\mathrm{T}}\boldsymbol{\beta}_{ijk} \\
&= \frac{1}{n\rho_n^{3/2}}\sum_{s=1}^d\sum_{j=1}^n\sum_{a=1}^n z_{iksja} + \frac{1}{n\rho_n}\sum_{s=1}^d\sum_{j=1}^n (A_{ij} - \rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})\beta_{ijks}\widetilde{R}_{js},
\end{aligned}$$

where $z_{iksja} = \zeta_{ias}\beta_{ijks}(A_{ij} - \rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})(A_{ja} - \rho_n\mathbf{x}_{0j}^{\mathrm{T}}\mathbf{x}_{0a})$. Clearly,

$$\frac{1}{n^2\rho_n^3}\max_{i\in[n]}\mathbb{E}_0\left\{\left(\sum_{s=1}^d\sum_{j=1}^n\sum_{a=1}^n z_{iksja}\right)^2\right\} \lesssim \frac{1}{n^2\rho_n^3}\sum_{s=1}^d\sum_{j=1}^n\sum_{a=1}^n\sum_{h=1}^n\sum_{b=1}^n\max_{i\in[n]}\mathbb{E}_0(z_{iksja}z_{ikshb}).$$

We now argue that the summation $\sum_{j=1}^n\sum_{a=1}^n\sum_{h=1}^n\sum_{b=1}^n\max_{i\in[n]}\max_{i\in[n]}\mathbb{E}_0(z_{iksja}z_{ikshb})$ is upper bounded by $\sup_{i,j}\|\boldsymbol{\zeta}_{ij}\|^2 n^2\rho_n$ up to a multiplicative constant. Note that as the indices j, a, h, b ranging over $[n]$, $\mathbb{E}_0(z_{iksja}z_{ikshb})$ is nonzero only if the cardinality of the collection of random variables $\{A_{ij}, A_{ih}, A_{aj}, A_{bh}\}$ is 2 or 4. These cases occur only if either one of the following cases happens:

1. A_{ij} and A_{ih} are the same random variable, and A_{aj}, A_{bh} are the same random variable. This happens only if one the following cases occur:

   (a) $(i, j) = (i, h), (a, j) = (b, h) \Rightarrow j = h$, $a = b$, and the number of terms is $O(n^2)$;

   (b) $(i, j) = (h, i), (a, j) = (b, h) \Rightarrow i = j = h$, $a = b$, and the number of terms is $O(n)$;

   (c) $(i, j) = (i, h), (a, j) = (h, b) \Rightarrow j = h = a = b$, and the number of terms is $O(n)$;

   (d) $(i, j) = (h, i), (a, j) = (h, b) \Rightarrow i = j = h = a = b$, and the number of terms is 1;

2. A_{ij} and A_{aj} are the same random variable, and A_{ih}, A_{bh} are the same random variable. This happens only if one the following cases occur:

   (a) $(i, j) = (a, j), (i, h) = (b, h) \Rightarrow i = a = b$, and the number of terms is $O(n^2)$;

   (b) $(i, j) = (j, a), (i, h) = (b, h) \Rightarrow i = j = a = b$, and the number of terms is $O(n)$;

   (c) $(i, j) = (a, j), (i, h) = (h, b) \Rightarrow i = h = a = b$, and the number of terms is $O(n)$;

   (d) $(i, j) = (j, a), (i, h) = (h, b) \Rightarrow i = j = h = a = b$, and the number of terms is 1;

3. A_{ij} and A_{bh} are the same random variable, and A_{ih}, A_{aj} are the same random variable. This happens only if one the following cases occur:

(a) $(i,j)=(b,h),(i,h)=(a,j)\Rightarrow i=b=a,\ h=j$, and the number of terms is $O(n)$;

(b) $(i,j)=(h,b),(i,h)=(a,j)\Rightarrow i=j=h=a=b$, and the number of terms is 1;

(c) $(i,j)=(b,h),(i,h)=(j,a)\Rightarrow j=j=h=a=b$, and the number of terms is 1;

(d) $(i,j)=(h,b),(i,h)=(j,a)\Rightarrow i=j=h=a=b$, and the number of terms is 1.

Therefore, the number of nonzero terms in the summation

$$\sum_{j=1}^n\sum_{a=1}^n\sum_{h=1}^n\sum_{b=1}^n\max_{i\in[n]}\max_{i\in[n]}\mathbb{E}_0(z_{iksja}z_{ikshb})$$

is $O(n^2)$. Furthermore, the centered second and fourth moments of Bernoulli$(\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})$ is upper bounded by ρ_n. Therefore, we obtain that

$$\frac{1}{n^2\rho_n^3}\max_{i\in[n]}\mathbb{E}_0\left\{\left(\sum_{s=1}^d\sum_{j=1}^n\sum_{a=1}^n z_{iksja}\right)^2\right\}\lesssim\frac{1}{n^2\rho_n^3}\sup_{i,j\in[n]}\|\boldsymbol{\zeta}_{ij}\|^2n^2\rho_n\lesssim\frac{1}{(n\rho_n)^2}.$$

In addition,

$$\begin{aligned}&\max_{i\in[n]}\left|\frac{1}{n\rho_n}\sum_{s=1}^d\sum_{j=1}^n(A_{ij}-\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})\beta_{ijks}R_{js}\right|\\&\quad\leq\frac{1}{n\rho_n}\max_{i\in[n]}\left\{\sum_{s=1}^d\sum_{j=1}^n(A_{ij}-\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})^2\beta_{jks}^2\right\}^{1/2}\|\widetilde{\mathbf{R}}\|_{\mathrm{F}}=O_{\mathbb{P}_0}\left(\frac{(\log n)^{\omega/2}}{n\rho_n^{3/2}}\right).\end{aligned}$$

Namely, this implies that for each individual $i\in[n]$,

$$\left|\frac{1}{n\sqrt{\rho_n}}\sum_{j=1}^n(A_{ij}-\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})(\rho_n^{-1/2}\mathbf{W}^{\mathrm{T}}\widetilde{\mathbf{x}}_j-\mathbf{x}_{0j})^{\mathrm{T}}\boldsymbol{\beta}_{ijk}\right|=O_{\mathbb{P}_0}\left(\frac{(\log n)^{\omega/2}}{n\rho_n^{3/2}}\right)$$

Furthermore,

$$\sum_{i=1}^n\mathbb{E}_0\left\{\left(\frac{1}{n\rho_n^{3/2}}\sum_{s=1}^d\sum_{j=1}^n\sum_{a=1}^n z_{iksja}\right)^2\right\}\leq\frac{1}{n\rho_n^3}\max_{i\in[n]}\mathbb{E}_0\left\{\left(\sum_{s=1}^d\sum_{j=1}^n\sum_{a=1}^n z_{iksja}\right)^2\right\}\lesssim\frac{1}{n\rho_n^2},$$

implying that

$$\sum_{i=1}^{n}\left\{\frac{1}{n\rho_n^{3/2}}\sum_{s=1}^{d}\sum_{j=1}^{n}\sum_{a=1}^{n}z_{iksja}\right\}^2 = O_{\mathbb{P}_0}\left(\frac{1}{n\rho_n^2}\right)$$

by Markov's inequality. Therefore, we conclude that

$$\begin{aligned}
&\sum_{i=1}^{n}\left\{\frac{1}{n\sqrt{\rho_n}}\sum_{j=1}^{n}(A_{ij}-\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})(\rho_n^{-1/2}\mathbf{W}^{\mathrm{T}}\widetilde{\mathbf{x}}_j-\mathbf{x}_{0j})^{\mathrm{T}}\boldsymbol{\beta}_{ijk}\right\}^2\\
&\quad\leq 2\sum_{i=1}^{n}\left\{\frac{1}{n\rho_n^{3/2}}\sum_{s=1}^{d}\sum_{j=1}^{n}\sum_{a=1}^{n}z_{iksja}\right\}^2 + 2n\max_{i\in[n]}\left|\frac{1}{n\rho_n}\sum_{s=1}^{d}\sum_{j=1}^{n}(A_{ij}-\rho_n\mathbf{x}_{0i}^{\mathrm{T}}\mathbf{x}_{0j})\beta_{ijks}R_{js}\right|^2\\
&\quad= O_{\mathbb{P}_0}\left(\frac{(\log n)^{\omega}}{n\rho_n^3}\right).
\end{aligned}$$

The proof is thus completed. □

Proof of Lemma 28. The proof of Lemma 30 relies on the following logarithmic Sobolev concentration inequality:

Lemma 30 (Theorem 6.7 in [69]) *Let $\mathbf{A},\mathbf{A}'\in\{0,1\}^{n\times n}$ be two symmetric hollow random adjacency matrices, $Z = Z(\mathbf{A})$ be a measurable function of of $\mathbf{A}$. Denote by $\mathbf{A}^{(kl)}$ the adjacency matrix obtained by replacing the (k,l) and (l,k) entries of $\mathbf{A}$ by those of $\mathbf{A}'$, and $Z_{kl} = Z(\mathbf{A}^{(kl)})$. If there exists a constant $v>0$ such that*

$$\mathbb{P}\left(\sum_{k<l}(Z-Z_{kl})>v\right)\leq\eta,$$

then for all $\epsilon>0$, $\mathbb{P}(|Z-\mathbb{E}(Z)|>t)\leq 2\exp\{-t^2/(2v)\}+\eta$.

Let $\mathbf{A}'$ be another symmetric hollow random adjacency matrix. Denote by $\mathbf{A}^{(kl)}$ the adjacency matrix obtained by replacing the (k,l) and (l,k) entries of $\mathbf{A}$ by those of $\mathbf{A}'$, and $Z_{kl} = Z(\mathbf{A}^{(kl)})$ Since that $\mathbf{A}$ and $\mathbf{A}^{(kl)}$ only differs by the (k,l) and (l,k) entries, and that the entries of $\mathbf{A}$ and $\mathbf{A}'$ are binary, we see that when $Z - Z_{kl}\neq 0$,

$$(A_{kl}-A_{kl}')(Z-Z_{kl}) = C_{1kl}+C_{2kl}+c_{kl},$$

where

$$C_{1kl} = 2\sum_{a=1}^{n}(A_{ka}-\rho_n\mathbf{x}_{0k}^{\mathrm{T}}\mathbf{x}_{0a})\boldsymbol{\gamma}_{kl}^{\mathrm{T}}\boldsymbol{\gamma}_{ka},\quad C_{2kl} = 2\sum_{a=1}^{n}(A_{la}-\rho_n\mathbf{x}_{0l}^{\mathrm{T}}\mathbf{x}_{0a})\boldsymbol{\gamma}_{lk}^{\mathrm{T}}\boldsymbol{\gamma}_{la},$$

and $c_{kl} = (1 - 2\rho_n \mathbf{x}_{0k}^{\mathrm{T}}\mathbf{x}_{0l})(\|\boldsymbol{\gamma}_{kl}\|^2 + \|\boldsymbol{\gamma}_{lk}\|^2) - 2(A_{kl} - \rho_n \mathbf{x}_{0k}^{\mathrm{T}}\mathbf{x}_{0l})\|\boldsymbol{\gamma}_{kl}\|^2 - 2(A_{lk} - \rho_n \mathbf{x}_{0l}^{\mathrm{T}}\mathbf{x}_{0k})\|\boldsymbol{\gamma}_{lk}\|^2$. Since

$$
\begin{aligned}
\sum_{k<l}\mathbb{E}_0(C_{1kl}^2) &= 4\sum_{k<l}\sum_{a=1}^n\sum_{b=1}^n \mathbb{E}_0\{(A_{ka} - \rho_n\mathbf{x}_{0k}^{\mathrm{T}}\mathbf{x}_{0a})(A_{kb} - \rho_n\mathbf{x}_{0k}^{\mathrm{T}}\mathbf{x}_{0b})\}(\boldsymbol{\gamma}_{kl}^{\mathrm{T}}\boldsymbol{\gamma}_{ka})(\boldsymbol{\gamma}_{kl}^{\mathrm{T}}\boldsymbol{\gamma}_{kb})\\
&= 4\sum_{k<l}\sum_{a=1}^n \mathbb{E}_0\{(A_{ka} - \rho_n\mathbf{x}_{0k}^{\mathrm{T}}\mathbf{x}_{0a})^2\}(\boldsymbol{\gamma}_{kl}^{\mathrm{T}}\boldsymbol{\gamma}_{ka})^2\\
&\leq 4\sum_{k<l}\sum_{a=1}^n \rho_n\mathbf{x}_{0k}^{\mathrm{T}}\mathbf{x}_{0a}(1 - \rho_n\mathbf{x}_{0k}^{\mathrm{T}}\mathbf{x}_{0a})\|\boldsymbol{\gamma}_{kl}\|^2\|\boldsymbol{\gamma}_{ka}\|^2 \lesssim \frac{1}{n\rho_n},\\
\sum_{k<l}\mathbb{E}_0(C_{2kl}^2) &= 4\sum_{k<l}\sum_{a=1}^n\sum_{b=1}^n \mathbb{E}_0\{(A_{la} - \rho_n\mathbf{x}_{0l}^{\mathrm{T}}\mathbf{x}_{0a})(A_{lb} - \rho_n\mathbf{x}_{0l}^{\mathrm{T}}\mathbf{x}_{0b})\}(\boldsymbol{\gamma}_{lk}^{\mathrm{T}}\boldsymbol{\gamma}_{la})(\boldsymbol{\gamma}_{lk}^{\mathrm{T}}\boldsymbol{\gamma}_{lb})\\
&= 4\sum_{k<l}\sum_{a=1}^n \mathbb{E}_0\{(A_{la} - \rho_n\mathbf{x}_{0l}^{\mathrm{T}}\mathbf{x}_{0a})^2\}(\boldsymbol{\gamma}_{lk}^{\mathrm{T}}\boldsymbol{\gamma}_{la})^2 \lesssim \frac{1}{n\rho_n},\\
\sum_{k<l}\mathbb{E}_0(c_{kl}^2) &\leq 6\sum_{k<l}(1 - 2\rho_n\mathbf{x}_{0k}^{\mathrm{T}}\mathbf{x}_{0l})(\|\boldsymbol{\gamma}_{kl}\|^4 + \|\boldsymbol{\gamma}_{lk}\|^4)\\
&\quad + 6\sum_{k<l}\mathbb{E}_0\{(A_{kl} - \rho_n\mathbf{x}_{0k}^{\mathrm{T}}\mathbf{x}_{0l})^2\}(\|\boldsymbol{\gamma}_{kl}\|^4 + \|\boldsymbol{\gamma}_{lk}\|^4)\\
&\lesssim \frac{1}{n^2\rho_n^2} + \frac{\rho_n}{n^2\rho_n^2} \lesssim \frac{1}{n^2\rho_n},
\end{aligned}
$$

we conclude that $\mathbb{E}_0\{\sum_{k<l}(Z - Z_{kl})^2\} \leq C/(n\rho_n)$ for some constant $C > 0$. Therefore, by Markov's inequality,

$$
\mathbb{P}\left(\sum_{k<l}(Z - Z_{kl})^2 > \frac{1}{\log n}\right) \leq \frac{C\log n}{n\rho_n} \leq \frac{C(\log n)^2}{n\rho_n^5} \to 0.
$$

Invoking Lemma 30, we obtain that

$$
\begin{aligned}
\mathbb{P}_0\left(|Z - \mathbb{E}_0(Z)| > \epsilon\right) &= \mathbb{P}_0\left[\left|Z - \frac{1}{n}\sum_{i=1}^n \operatorname{tr}\{\mathbf{G}_n(\mathbf{x}_{0i})^{-1}\}\right| > \epsilon\right]\\
&\leq 2\exp\left(-\frac{1}{2}\epsilon^2\log n\right) + \frac{C\log n}{n\rho_n} \to 0
\end{aligned}
$$

for all $\epsilon > 0$. The proof is thus completed. $\square$

Proof of Lemma 29. We first show that $\mathbf{G}_n(\mathbf{x}) \to \mathbf{G}(\mathbf{x})$ as $n \to \infty$ uniformly for all $\mathbf{x} \in \mathcal{X}(\delta)$. It suffices to show that for all $s, t \in [d]$,

$$
\sup_{\mathbf{u}\in\mathcal{X}(\delta)}\left|\int \frac{x_s x_t}{\mathbf{x}^{\mathrm{T}}\mathbf{u}(1 - \rho_n\mathbf{x}^{\mathrm{T}}\mathbf{u})}F_n(\mathrm{d}\mathbf{x}) - \int_{\mathcal{X}(\delta)} \frac{x_s x_t}{\mathbf{x}^{\mathrm{T}}\mathbf{u}(1 - \rho_n\mathbf{x}^{\mathrm{T}}\mathbf{u})}F(\mathrm{d}\mathbf{x})\right| \to 0
$$

as $n \to \infty$, where $\mathbf{x} = [x_1, \ldots, x_d]^{\mathrm{T}}$ and $\mathbf{u} = [u_1, \ldots, u_d]^{\mathrm{T}}$. By the multivariate integration by parts, we have,

$$\begin{aligned}
&\left|\int \frac{x_s x_t}{\mathbf{x}^{\mathrm{T}}\mathbf{u}(1-\rho_n \mathbf{x}^{\mathrm{T}}\mathbf{u})} F_n(\mathrm{d}\mathbf{x}) - \int_{\mathcal{X}(\delta)} \frac{x_s x_t}{\mathbf{x}^{\mathrm{T}}\mathbf{u}(1-\rho_n \mathbf{x}^{\mathrm{T}}\mathbf{u})} F(\mathrm{d}\mathbf{x})\right| \\
&\quad \le \sup_{\mathbf{x}\in\mathcal{X}} |F_n(\mathbf{x}) - F(\mathbf{x})| \left|\int \frac{\partial^d}{\partial x_1 \ldots \partial x_d}\left\{\frac{x_s x_t}{\mathbf{x}^{\mathrm{T}}\mathbf{u}(1-\rho_n \mathbf{x}^{\mathrm{T}}\mathbf{u})}\right\} \mathrm{d}\mathbf{x}\right| \\
&\quad \le \sup_{\mathbf{x}\in\mathcal{X}} |F_n(\mathbf{x}) - F(\mathbf{x})| \int \left|\frac{\partial^d}{\partial x_1 \ldots \partial x_d}\left\{\frac{x_s x_t}{\mathbf{x}^{\mathrm{T}}\mathbf{u}(1-\rho_n \mathbf{x}^{\mathrm{T}}\mathbf{u})}\right\}\right| \mathrm{d}\mathbf{x}.
\end{aligned}$$

Therefore, it is in turn sufficient to show that for all $s, t \in [d]$,

$$\sup_{\mathbf{x},\mathbf{u}\in\mathcal{X}(\delta)} \left|\frac{\partial^d}{\partial x_1 \ldots \partial x_d}\left\{\frac{x_s x_t}{\mathbf{x}^{\mathrm{T}}\mathbf{u}(1-\rho_n \mathbf{x}^{\mathrm{T}}\mathbf{u})}\right\}\right| < \infty.$$

The cases where $d = 1$ and $d = 2$ are trivial and we assume that $d \ge 3$. Let us first consider the case where $s \neq t$, and without loss of generality we may also assume that $s = d-1$, $t = d$. Denote $f(y) = 1/y$, $g_n(\mathbf{x}, \mathbf{u}) = \mathbf{u}^{\mathrm{T}}\mathbf{x}(1-\rho_n \mathbf{u}^{\mathrm{T}}\mathbf{x})$. By the multivariate Faà di Bruno's formula [94],

$$h_n(\mathbf{x}, \mathbf{u}) = \frac{\partial^{d-2} f(g_n(\mathbf{x}, \mathbf{u}))}{\partial x_1 \ldots \partial x_{d-2}} = \sum_{\pi\in\Pi} f^{(|\pi|)}(y) \prod_{B\in\pi} \frac{\partial^{|B|} g_n(\mathbf{x}, \mathbf{u})}{\prod_{j\in B} \partial x_j},$$

where $\pi \in \Pi$ ranges over the set of all partitions of $[d-2]$, $B \in \pi$ ranges over all sets in the partition π, $|\pi|$ is the number of sets in the partition π, and $|B|$ is the cardinality of the set B. Clearly,

$$\frac{\partial^d \{x_{d-1} x_d f(g_n(\mathbf{x}, \mathbf{u}))\}}{\partial x_1 \ldots \partial x_d} = x_{d-1} \frac{\partial h_n(\mathbf{x}, \mathbf{u})}{\partial x_d} + x_{d-1} \frac{\partial h_n(\mathbf{x}, \mathbf{u})}{\partial x_{d-1}} + x_{d-1} x_d \frac{\partial^2 h_n(\mathbf{x}, \mathbf{u})}{\partial x_{d-1} \partial x_d}.$$

Directly computation of derivatives yields

$$\begin{aligned}
\frac{\partial h_n(\mathbf{x}, \mathbf{u})}{\partial x_{d-1}} &= \sum_{\pi\in\Pi} \{D_{\pi 1}(\mathbf{x}, \mathbf{u}) + D_{\pi 2}(\mathbf{x}, \mathbf{u})\}, \\
D_{\pi 1}(\mathbf{x}, \mathbf{u}) &= f^{(|\pi|+1)}(y) \frac{\partial g_n(\mathbf{x}, \mathbf{u})}{\partial x_{d-1}} \prod_{B\in\pi} \frac{\partial^{|B|} g_n(\mathbf{x}, \mathbf{u})}{\prod_{j\in B} \partial x_j}, \\
D_{\pi 2}(\mathbf{x}, \mathbf{u}) &= f^{(|\pi|)}(y) \sum_{B\in\pi} \frac{\partial^{|B|+1} g_n(\mathbf{x}, bu)}{\partial x_{d+1} \prod_{j\in B} \partial x_j} \prod_{C\in\pi\setminus\{B\}} \frac{\partial^{|C|} g_n(\mathbf{x}, \mathbf{u})}{\prod_{j\in C} \partial x_j}, \\
\frac{\partial D_{\pi 1}(\mathbf{x})}{\partial x_d} &= f^{(|\pi|+2)}(y) \frac{\partial g_n(\mathbf{x}, \mathbf{u})}{\partial x_d} \frac{\partial g_n(\mathbf{x}, \mathbf{u})}{\partial x_{d-1}} \prod_{B\in\pi} \frac{\partial^{|B|} g_n(\mathbf{x}, \mathbf{u})}{\prod_{j\in B} \partial x_j}
\end{aligned}$$

$$
\begin{aligned}
&+ f^{(|\pi|+1)}(y)\frac{\partial^2 g_n(\mathbf{x},\mathbf{u})}{\partial x_d \partial x_{d-1}} \prod_{B\in\pi} \frac{\partial^{|B|} g_n(\mathbf{x},\mathbf{u})}{\prod_{j\in B}\partial x_j} \\
&+ f^{(|\pi|+1)}(y)\frac{\partial g_n(\mathbf{x},\mathbf{u})}{\partial x_{d-1}} \sum_{B\in\pi} \frac{\partial^{|B|+1} g_n(\mathbf{x},\mathbf{u})}{\partial x_d \prod_{j\in B}\partial x_j} \prod_{C\in\pi\setminus\{B\}} \frac{\partial^{|C|} g_n(\mathbf{x},\mathbf{u})}{\prod_{j\in C}\partial x_j},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial D_{\pi 2}(\mathbf{x})}{\partial x_d} &= f^{(|\pi|+1)}(y)\frac{\partial g_n(\mathbf{x},\mathbf{u})}{\partial x_d} \sum_{B\in\pi} \frac{\partial^{|B|+1} g_n(\mathbf{x},\mathbf{u})}{\partial x_d \prod_{j\in B}\partial x_j} \prod_{C\in\pi\setminus\{B\}} \frac{\partial^{|C|} g_n(\mathbf{x},\mathbf{u})}{\prod_{j\in C}\partial x_j} \\
&+ f^{(|\pi|)}(y) \sum_{B\in\pi} \frac{\partial^{|B|+2} g_n(\mathbf{x},\mathbf{u})}{\partial x_d \partial x_{d-1} \prod_{j\in C}\partial x_j} \prod_{C\in\pi\setminus\{B\}} \frac{\partial^{|C|} g_n(\mathbf{x},\mathbf{u})}{\prod_{j\in C}\partial x_j} \\
&+ f^{(|\pi|)}(y) \sum_{B\in\pi} \frac{\partial^{|B|+1} g_n(\mathbf{x},\mathbf{u})}{\partial x_{d-1} \prod_{j\in C}\partial x_j} \\
&\times \left\{ \sum_{C\in\pi\setminus\{B\}} \frac{\partial^{|C|+1} g_n(\mathbf{x},\mathbf{u})}{\partial x_d \prod_{j\in C}\partial x_j} \prod_{D\in\pi\setminus\{B,C\}} \frac{\partial^{|D|} g_n(\mathbf{x},\mathbf{u})}{\prod_{j\in D}\partial x_j} \right\}.
\end{aligned}
$$

Note that for any finite $t \in \mathbb{N}_+$,

$$
\sup_{\mathbf{x},\mathbf{u}\in\mathcal{X}(\delta)} |f^{(t)}(y)| = t! \sup_{\mathbf{x},\mathbf{u}\in\mathcal{X}(\delta)} \{\mathbf{x}^{\mathrm{T}}\mathbf{u}(1-\rho_n \mathbf{x}^{\mathrm{T}}\mathbf{u})\}^{-(t+1)} \le \frac{t!}{\delta^{2(t+1)}} < \infty,
$$

and for any $k, l \in [d]$,

$$
\begin{aligned}
&\sup_{n\ge 1,\mathbf{x},\mathbf{u}\in\mathcal{X}(\delta)} \left|\frac{\partial g_n(\mathbf{x},\mathbf{u})}{\partial x_k}\right| = |u_k - 2\rho_n(\mathbf{u}^{\mathrm{T}}\mathbf{x})u_k| \le 3, \\
&\sup_{n\ge 1,\mathbf{x},\mathbf{u}\in\mathcal{X}(\delta)} \left|\frac{\partial^2 g_n(\mathbf{x},\mathbf{u})}{\partial x_k \partial x_l}\right| = |-2\rho_n u_k u_l| \le 3, \quad \sup_{n\ge 1,\mathbf{x},\mathbf{u}\in\mathcal{X}(\delta)} \left|\frac{\partial^2 g_n(\mathbf{x},\mathbf{u})}{\partial x_k \partial x_l}\right| = 0.
\end{aligned}
$$

Therefore, by the fact that the summations $\sum_{B\in\pi}$ and $\sum_{\pi\in\Pi}$ have finitely many terms, we see immediately that

$$
\begin{aligned}
&\sup_{n\ge 1,\mathbf{x},\mathbf{u}\in\mathcal{X}(\delta)} |h_n(\mathbf{x},\mathbf{u})| < \infty, \quad \sup_{n\ge 1,\mathbf{x},\mathbf{u}\in\mathcal{X}(\delta)} |D_{\pi_1}(\mathbf{x},\mathbf{u})| < \infty, \\
&\sup_{n\ge 1,\mathbf{x},\mathbf{u}\in\mathcal{X}(\delta)} |D_{\pi_2}(\mathbf{x},\mathbf{u})| < \infty, \\
&\sup_{n\ge 1,\mathbf{x},\mathbf{u}\in\mathcal{X}(\delta)} \left|\frac{\partial}{\partial x_d} D_{\pi_1}(\mathbf{x},\mathbf{u})\right| < \infty, \quad \sup_{n\ge 1,\mathbf{x},\mathbf{u}\in\mathcal{X}(\delta)} \left|\frac{\partial}{\partial x_d} D_{\pi_2}(\mathbf{x},\mathbf{u})\right| < \infty, \\
&\sup_{n\ge 1,\mathbf{x},\mathbf{u}\in\mathcal{X}(\delta)} \left|\frac{\partial h_n(\mathbf{x},\mathbf{u})}{\partial x_{d-1}}\right| \le \sum_{\pi\in\Pi} \sup_{n\ge 1,\mathbf{x},\mathbf{u}\in\mathcal{X}(\delta)} |D_{\pi 1}(\mathbf{x},\mathbf{u})| + \sum_{\pi\in\Pi} \sup_{n\ge 1,\mathbf{x},\mathbf{u}\in\mathcal{X}(\delta)} |D_{\pi 2}(\mathbf{x},\mathbf{u})| < \infty, \\
&\sup_{n\ge 1,\mathbf{x},\mathbf{u}\in\mathcal{X}(\delta)} \left|\frac{\partial^2 h_n(\mathbf{x},\mathbf{u})}{\partial x_d \partial x_{d-1}}\right| < \infty.
\end{aligned}
$$

Hence we finish proving that

$$
\sup_{n\ge 1,\mathbf{x},\mathbf{u}\in\mathcal{X}(\delta)} \left|\frac{\partial^d}{\partial x_1 \ldots \partial x_d} \left\{\frac{x_s x_t}{\mathbf{u}^{\mathrm{T}}\mathbf{x}(1-\rho_n \mathbf{u}^{\mathrm{T}}\mathbf{x})}\right\}\right| < \infty
$$

when $s \neq t$. The proof for case where $s = t$ follows the exactly same lines as that for the case where $s \neq t$. Therefore, we conclude that $\sup_{\mathbf{x}\in\mathcal{X}(\delta)} \|\mathbf{G}_n(\mathbf{x}) - \mathbf{G}(\mathbf{x})\|_{\mathrm{F}} \to 0$.

We next show that $\sup_{\mathbf{x}\in\mathcal{X}(\delta)} \|\mathbf{G}_n(\mathbf{x})^{-1} - \mathbf{G}(\mathbf{x})^{-1}\|_{\mathrm{F}} \to 0$. This immediately follows from the inequality

$$\sup_{\mathbf{x}\in\mathcal{X}(\delta)} \|\mathbf{G}_n(\mathbf{x}) - \mathbf{G}(\mathbf{x})\|_{\mathrm{F}} \leq \sup_{\mathbf{x}\in\mathcal{X}(\delta)} \|\mathbf{G}_n(\mathbf{x})^{-1}\|_{\mathrm{F}} \|\mathbf{G}_n(\mathbf{x}) - \mathbf{G}(\mathbf{x})\|_{\mathrm{F}} \|\mathbf{G}(\mathbf{x})^{-1}\|_{\mathrm{F}},$$

the fact that $\mathbf{G}_n(\mathbf{x}) \succeq (1/2)\boldsymbol{\Delta}$ and $\mathbf{G}(\mathbf{x}) \succeq \boldsymbol{\Delta}$ for sufficiently large n, and the uniform convergence of $\mathbf{G}_n(\mathbf{x}) \to \mathbf{G}(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{X}(\delta)$.

We finally show that $\widetilde{\mathbf{G}}_n(\mathbf{x}) - \widetilde{\mathbf{G}}(\mathbf{x})$ uniformly for all $\mathbf{x} \in \mathcal{X}(\delta)$. This result also follows from the fact that

$$\begin{aligned}
\sup_{\mathbf{x}\in\mathcal{X}(\delta)} \left| \frac{1}{\boldsymbol{\mu}_n^{\mathrm{T}}\mathbf{x}} - \frac{1}{\boldsymbol{\mu}^{\mathrm{T}}\mathbf{x}} \right| &\leq \frac{1}{\delta^2} \|\boldsymbol{\mu}_n - \boldsymbol{\mu}\| \to 0, \\
\sup_{\mathbf{x}\in\mathcal{X}(\delta)} \left\| \frac{\mathbf{x}\boldsymbol{\mu}_n^{\mathrm{T}}}{(\boldsymbol{\mu}_n^{\mathrm{T}}\mathbf{x})^2} - \frac{\mathbf{x}\boldsymbol{\mu}^{\mathrm{T}}}{(\boldsymbol{\mu}^{\mathrm{T}}\mathbf{x})^2} \right\|_{\mathrm{F}} &\leq \frac{3}{\delta^4} \|\boldsymbol{\mu}_n - \boldsymbol{\mu}\| \to 0,
\end{aligned}$$

and the uniform convergence result $\sup_{\mathbf{x}\in\mathcal{X}(\delta)} \|\mathbf{G}_n(\mathbf{x})^{-1} - \mathbf{G}(\mathbf{x})^{-1}\|_{\mathrm{F}} \to 0$. The proof is thus completed. $\square$

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